NE150/215M Introduction to Nuclear Reactor Theory
Spring 2022

Discussion 6: Diffusion Equation

March 16th, 2022

Helpful Readings: LE Ch.6, LB Ch.5

Ian Kolaja

Warmup

1) A heterogenous reactor core is fueled with UO_2. The volumetric composition of the core is 45% fuel, 35% coolant and 20% structural material. The flux is uniform. Calculate the thermal utilization factor.

Material	$\Sigma_f\ (cm^{-1})$	$\Sigma_a\ (cm^{-1})$
UO_2	0.001	0.010
Na	$-$	0.00008
Fe	$-$	0.0007

Warmup

1) A heterogenous reactor core is fueled with UO_2. The volumetric composition of the core is 45% fuel, 35% coolant and 20% structural material. The flux is uniform. Calculate the thermal utilization factor.

$$f = \frac{therm.\,neutrons\ absorbed\ in\ fuel}{therm.\,neutrons\ absorbed\ in\ all\ materials}$$

Material	$\Sigma_f\ (cm^{-1})$	$\Sigma_a\ (cm^{-1})$
UO_2	0.001	0.010
Na	–	0.00008
Fe	–	0.0007

Warmup

1) A heterogenous reactor core is fueled with UO_2. The volumetric composition of the core is 45% fuel, 35% coolant and 20% structural material. The flux is uniform. Calculate the thermal utilization factor.

$$f = \frac{\Sigma_a^{fuel} V^{fuel} \phi^{fuel}}{\Sigma_a^{fuel} V^{fuel} \phi^{fuel} + \Sigma_a^{cool} V^{cool} \phi^{cool} + \Sigma_a^{strc} V^{strc} \phi^{strc}}$$

Material	$\Sigma_f\ (cm^{-1})$	$\Sigma_a\ (cm^{-1})$
UO_2	0.001	0.010
Na	–	0.00008
Fe	–	0.0007

Warmup

1) A heterogenous reactor core is fueled with UO_2. The volumetric composition of the core is 45% fuel, 35% coolant and 20% structural material. The flux is uniform. Calculate the thermal utilization factor.

$$f = \frac{\Sigma_a^{fuel} V^{fuel}}{\Sigma_a^{fuel} V^{fuel} + \Sigma_a^{cool} V^{cool} + \Sigma_a^{strc} V^{strc}}$$

Material	$\Sigma_f\ (cm^{-1})$	$\Sigma_a\ (cm^{-1})$
UO_2	0.001	0.010
Na	$-$	0.00008
Fe	$-$	0.0007

Warmup

1) A heterogenous reactor core is fueled with UO_2. The volumetric composition of the core is 45% fuel, 35% coolant and 20% structural material. The flux is uniform. Calculate the thermal utilization factor.

$$f = \frac{(0.010\ cm^{-1})(0.45)}{(0.010\ cm^{-1})(0.45) + (0.00008\ cm^{-1})(0.35) + (0.0007\ cm^{-1})(0.2)}$$

$$f = 0.964$$

Material	$\Sigma_f\ (cm^{-1})$	$\Sigma_a\ (cm^{-1})$
UO_2	0.001	0.010
Na	–	0.00008
Fe	–	0.0007

Logistics

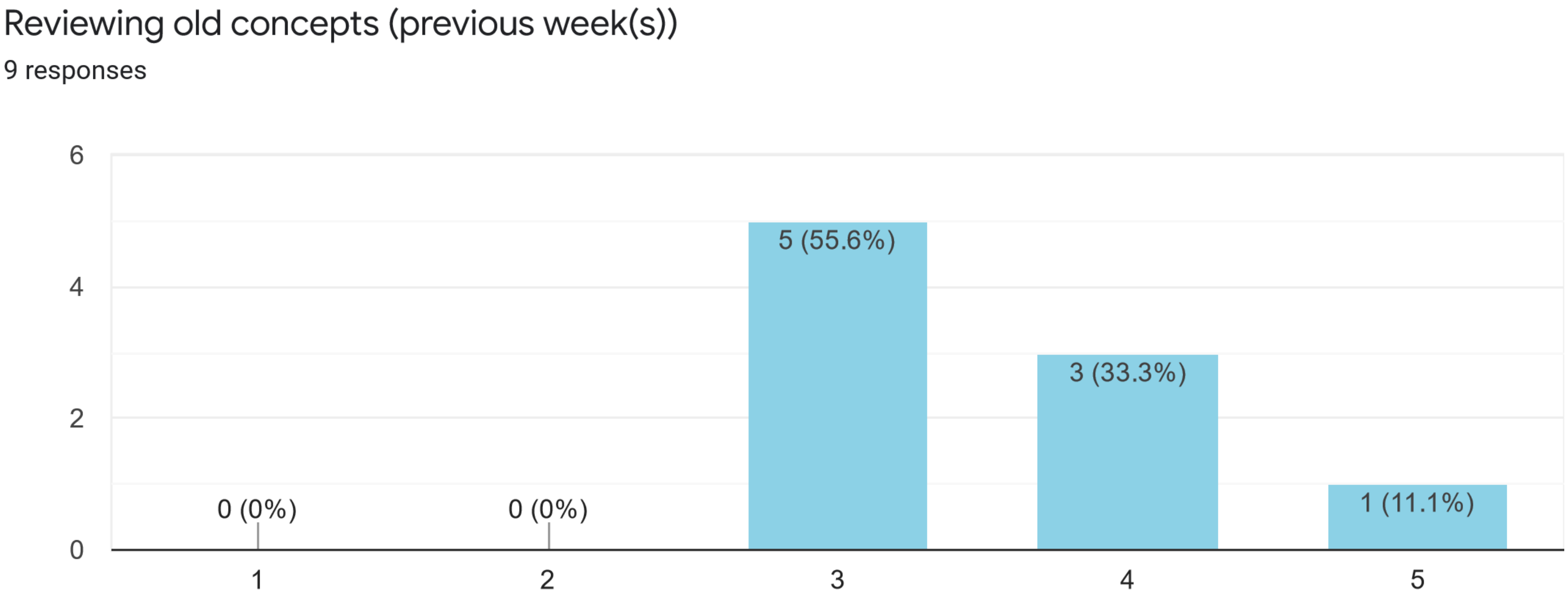
Reminder: Grading

I do not grade homework or exams!

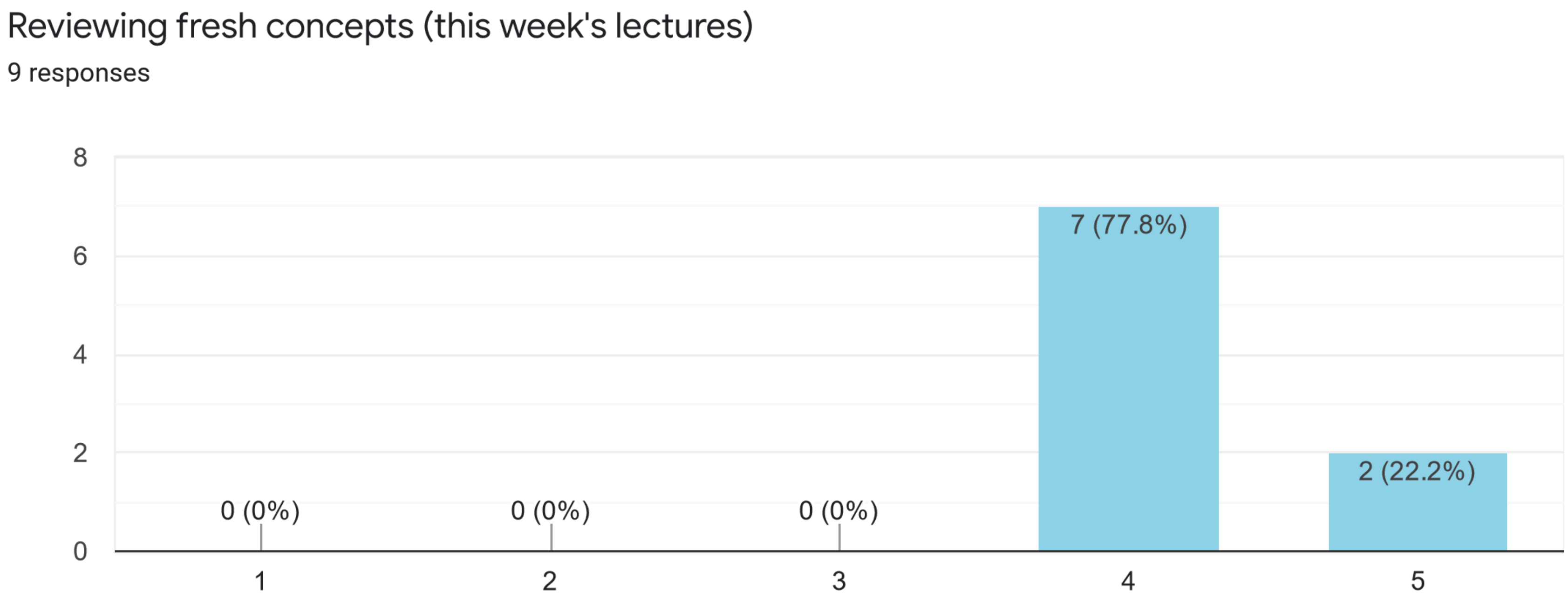
HW Grading: Evan Still evanstill@berkeley.edu

Exam Grading: Max

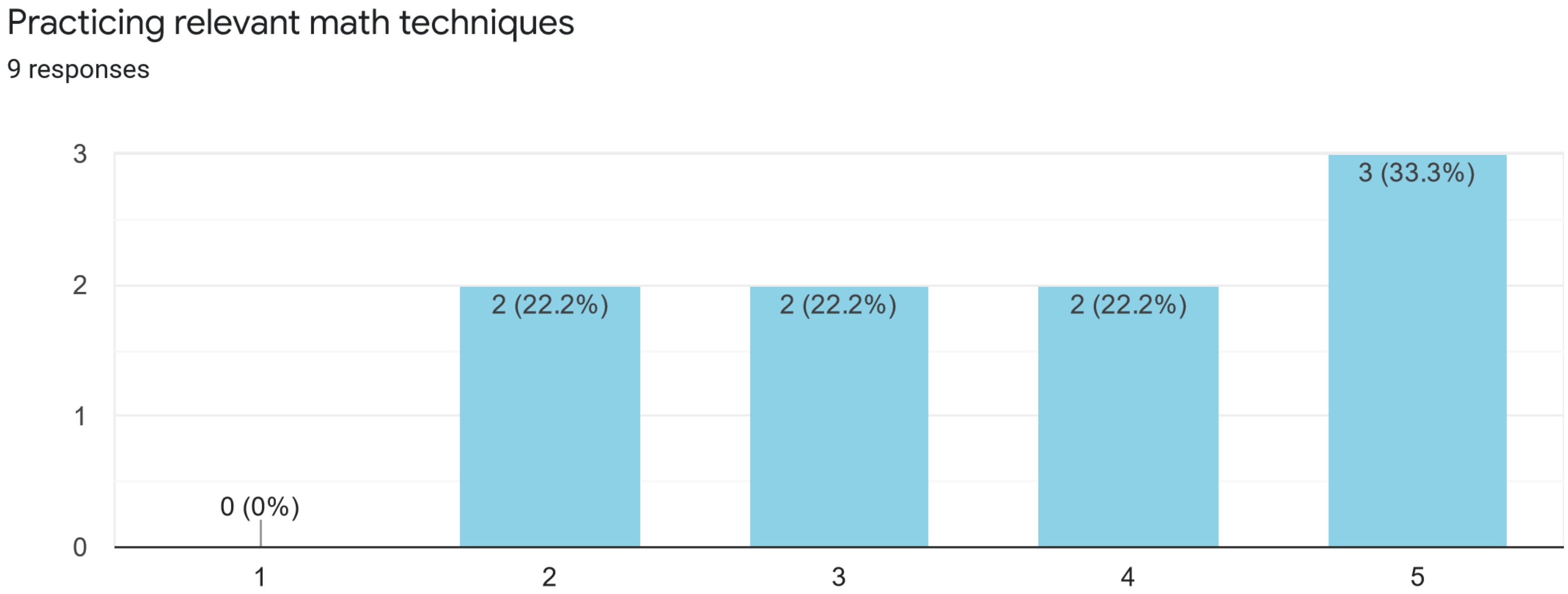
Survey results: Discussion Priorities



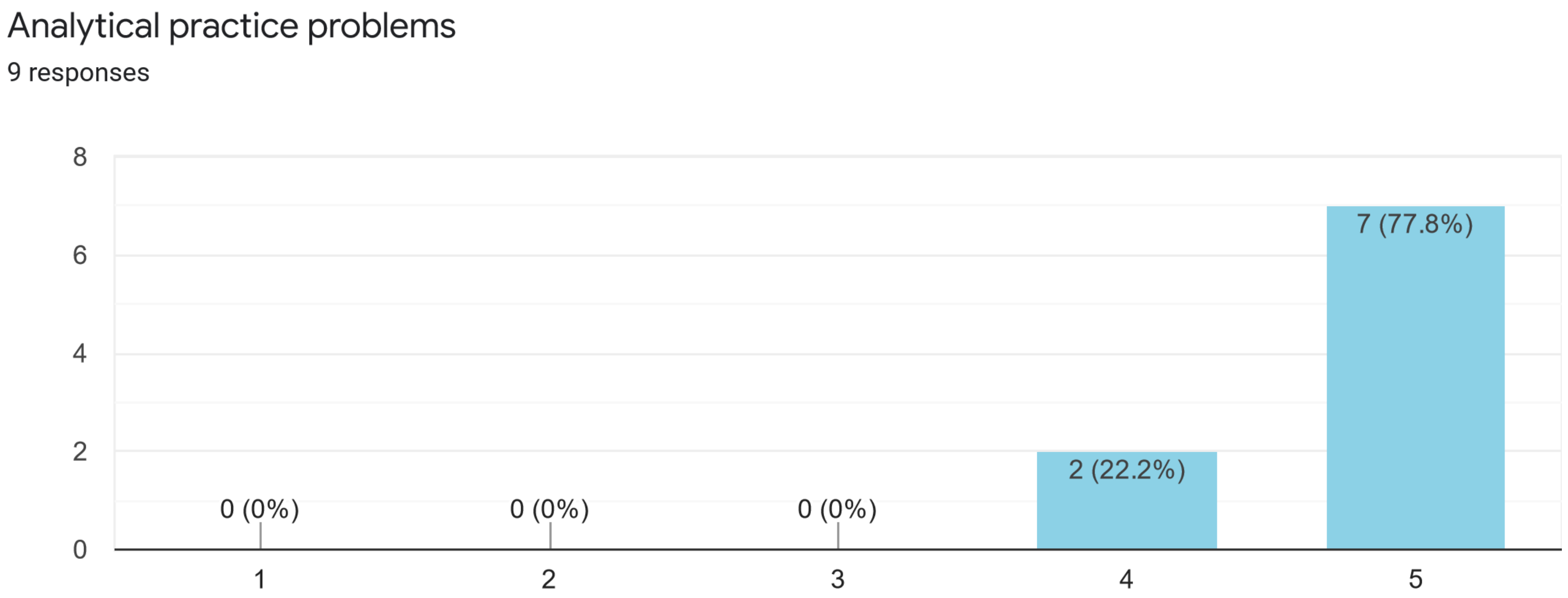
Survey results: Discussion Priorities



Survey results: Discussion Priorities



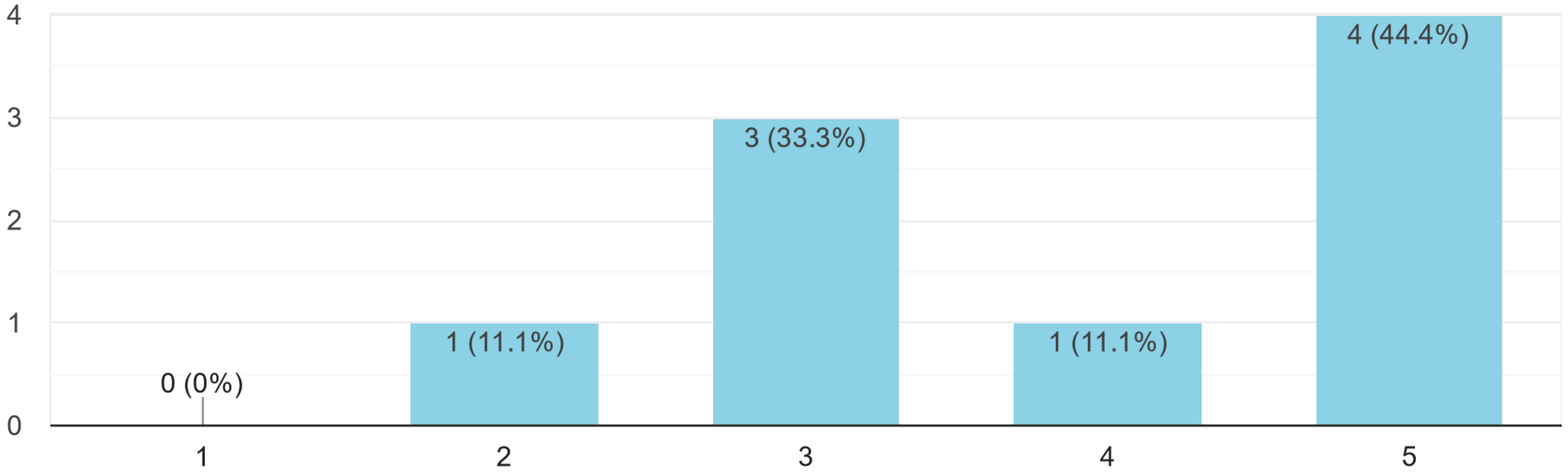
Survey results: Discussion Priorities



Survey results: Discussion Priorities

Conceptual practice problems / open ended discussions

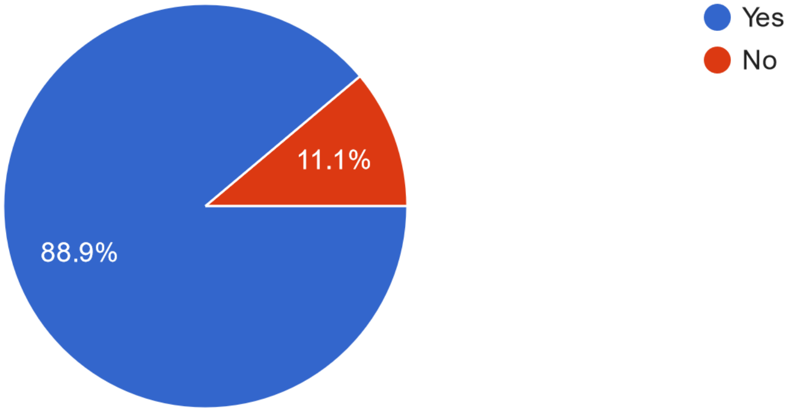
9 responses



Survey results: Serpent workshop

Would you attend an optional Serpent workshop if you were available at the time?

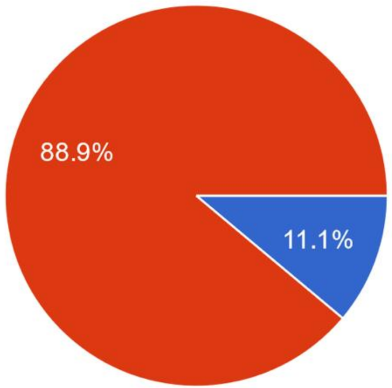
9 responses



Survey results: Office Hours

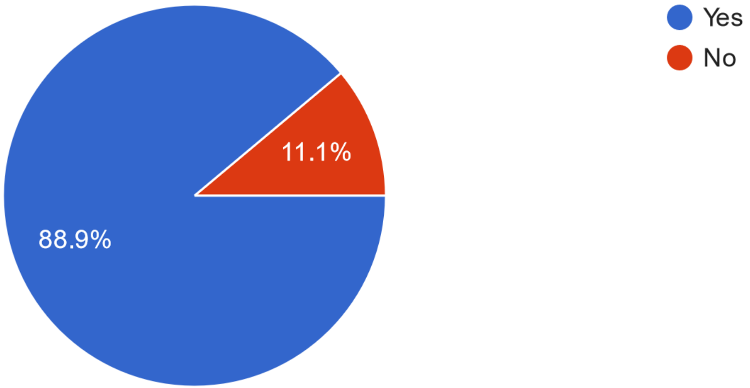
Would you like Ian's office hours rescheduled?

9 responses



Would you physically go to Ian's office hours if they were in person?

9 responses



Review

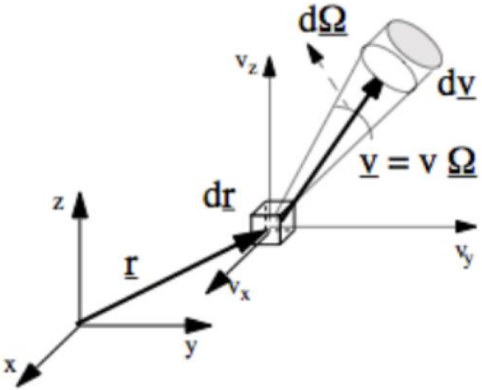
Last Time: Neutron Transport Equation

$$\underbrace{\frac{1}{v}\frac{\partial\psi}{\partial t}(\vec{r},E,\hat{\Omega},t)}_{\text{time rate of change}}+\underbrace{\hat{\Omega}\cdot\nabla\psi(\vec{r},E,\hat{\Omega},t)}_{\text{streaming loss rate}}+\underbrace{\Sigma_t(\vec{r},E)\psi(\vec{r},E,\hat{\Omega},t)}_{\text{total interaction loss rate}}$$

$$=\underbrace{\int_0^\infty\int_{4\pi}\Sigma_s(\vec{r},E'\to E,\hat{\Omega}'\to\hat{\Omega})\psi(\vec{r},E',\hat{\Omega}',t)d\hat{\Omega}'dE'}_{\text{in scattering source rate}}$$

$$+\underbrace{\frac{\chi_p(E)}{4\pi}\int_0^\infty\int_{4\pi}\nu(E')\Sigma_f(\vec{r},E')\psi(\vec{r},E',\hat{\Omega}',t)d\hat{\Omega}'dE'}_{\text{fission source rate}}$$

$$+\underbrace{S(\vec{r},E,\hat{\Omega},t)}_{\text{external source rate}}.$$



NE155 notes, Fall 2019

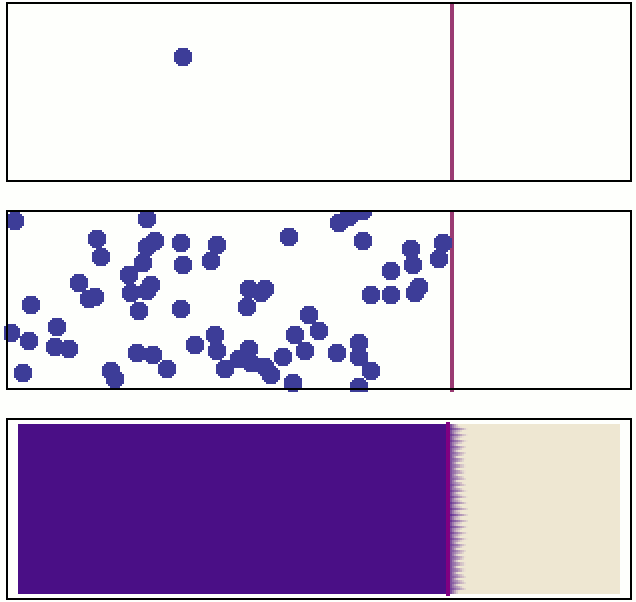
What if you don’t want to analytically solve the transport equation?

- I get it
- Also, this isn’t NE155
- We’ll be using the diffusion equation
- This approximation largely depends on neglecting the angular dependence of flux.
- Physically, this means that neutrons move with their concentration gradient as in **Fick’s Law**.

Fick’s First Law

- Statement that flux of diffusing species goes from regions of high concentration to regions of low concentration, proportional to concentration gradient.

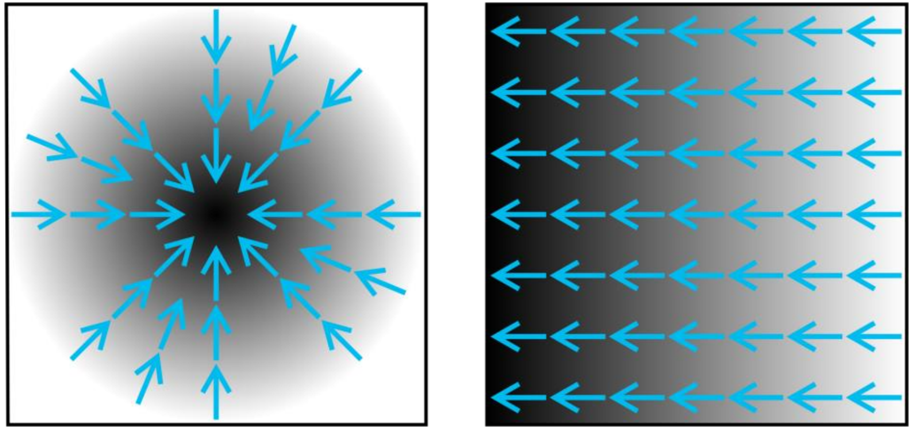
$$\vec{J} = -D\nabla\phi$$



Math Review: Gradient

Let f be a scalar-valued, differentiable function f of several variables $(x_1, \ldots x_n)$. The gradient ∇f at point p is the vector whose partial derivatives are given as below:

$$\nabla f(p) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(p) \\ \vdots \\ \frac{\partial f}{\partial x_n}(p) \end{bmatrix}$$



Diffusion Equation (Monoenergetic)

$$\frac{1}{v}\frac{\partial \phi(\vec{r},t)}{\partial t} = S(\vec{r},t) - \Sigma_a(\vec{r})\phi(\vec{r},t) + \nabla \cdot D(\vec{r})\nabla\phi(\vec{r},t)$$

Rate of Change Source Absorption Leakage

With one neutron energy.

$\phi(\vec{r},t)$, Scalar Flux $\left(\frac{\text{n}}{\text{cm}^2\text{s}}\right)$

$D(\vec{r})$, Diffusion Coefficient, (cm)

$S_{ext}(\vec{r},t)$, Independent source of neutrons $\left(\frac{\#}{\text{cm}^3\text{s}}\right)$

Diffusion Equation (Simplified)

$$0 = S(\vec{r}) - \Sigma_a \phi(\vec{r}) + D\nabla^2 \phi(\vec{r})$$

Rate of Change Source Absorption Leakage

Steady state, with one neutron energy, uniform.

Diffusion Length: $L^2 = \frac{D}{\Sigma_a}\ [cm^2]$

Math Review: Laplace operator

The Laplacian is the divergence of the gradient of a scalar function. Intuitively, the Laplacian $\Delta f(p)$ of a function f at point p tells you how much the average value of f over small spheres centered at p deviates from $f(p)$

$$\nabla \cdot \nabla f = \nabla^2 f = \mathrm{div}(\mathrm{grad}(\mathrm{f})) = \Delta \mathrm{f} = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

Math Review: Laplace operator

Geometry	$\Delta = \nabla^2 =$ (General)	$\Delta = \nabla^2 =$ (1D)
Cartesian	$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$	$\frac{\partial^2}{\partial x^2}$
Cylindrical	$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$	$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}$
Spherical	$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}$

Diffusion Equation Assumptions

Assumption 1) Scattering is isotropic in the LAB coordinate system

$$D = \frac{1}{3\Sigma_{tr}} = \frac{\lambda_{tr}}{3}, \qquad \Sigma_{tr} = \Sigma_s(1 - \bar{\mu}), \qquad where\ \mu = \cos\theta$$

Assumption 2) The scattering cross section is much higher than the absorption

Diffusion Equation Assumptions

Assumption 3) The medium is infinite

Assumption 4) Flux varies slowly with position

Diffusion Equation Applicability

The assumptions that go into the diffusion equation are valid when the solution is **not**:

1. Near a void
2. Near a boundary where material properties change rapidly
3. Near a localized source
4. In a strong absorber

Boundary Conditions

1) Initial Condition

Specifies the neutron flux for all positions at the initial time

$\phi(\vec{r}, t = 0) = \phi_0(\vec{r})$

2) Finite Flux

For flux to physically make sense, it must be real, non-negative, and finite. (Away from localized sources)

$0 \leq \phi(\vec{r}, t) < \infty$

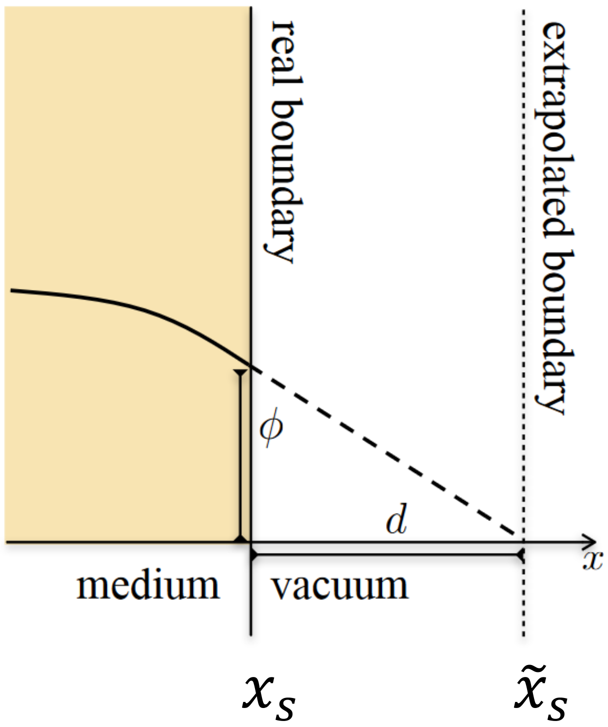
Boundary Conditions

3) Vacuum

Neutrons cannot enter the reactor from the outside; thus, inward directed partial current vanishes at reactor boundary

$$J^{-}(x_s) = \frac{1}{4}\phi(x_s) + \frac{D}{2}\frac{d\phi}{dx}\Big|_{x_s} = 0, \; or$$

$$\phi(\tilde{x}_s) = 0, \; where \; \tilde{x}_s = x_s + 2D$$



Boundary Conditions

4) Interface

The flux and normal component of the current are continuous at an interface $\vec{r}_s$ between two different materials.

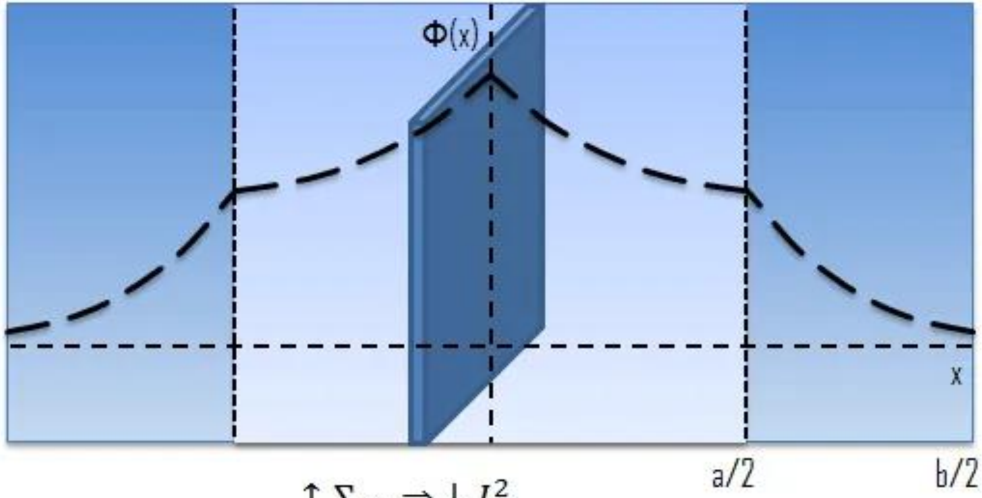
$$\phi_1(\vec{r}_s, t) = \phi_2(\vec{r}_s, t), \qquad J_1(\vec{r}_s, t) = J_2(\vec{r}_s, t)$$

$$-D_1 \nabla \phi_1(\vec{r}_s) = -D_2 \nabla \phi_2(\vec{r}_s)$$

Boundary Conditions

4) Interface

$L_2^2 = \frac{D_2}{\Sigma_{a2}}$ $\qquad$ $L_1^2 = \frac{D_1}{\Sigma_{a1}}$



$\uparrow \Sigma_{a2} \Rightarrow \downarrow L_2^2$

$L_1^2 > L_2^2$ $\qquad$ $\Rightarrow$ *steeper slope of* $\phi(x)$

$\uparrow \Sigma_{s2} \Rightarrow \downarrow L_2^2$

"Boundary Conditions – Diffusion Equation", nuclear-power.com

Boundary Conditions

5) Source

All of the neutrons flowing through the bounding area of a neutron source have to come from it.

$$S(x_0) = \lim_{x \to x_0} \int_S \vec{J} \cdot \hat{\ell}_s dS \ (General)$$

Boundary Conditions

5) Source

$$\text{Planar Source: } \lim_{x \to 0} J(x) = \frac{S}{2}$$

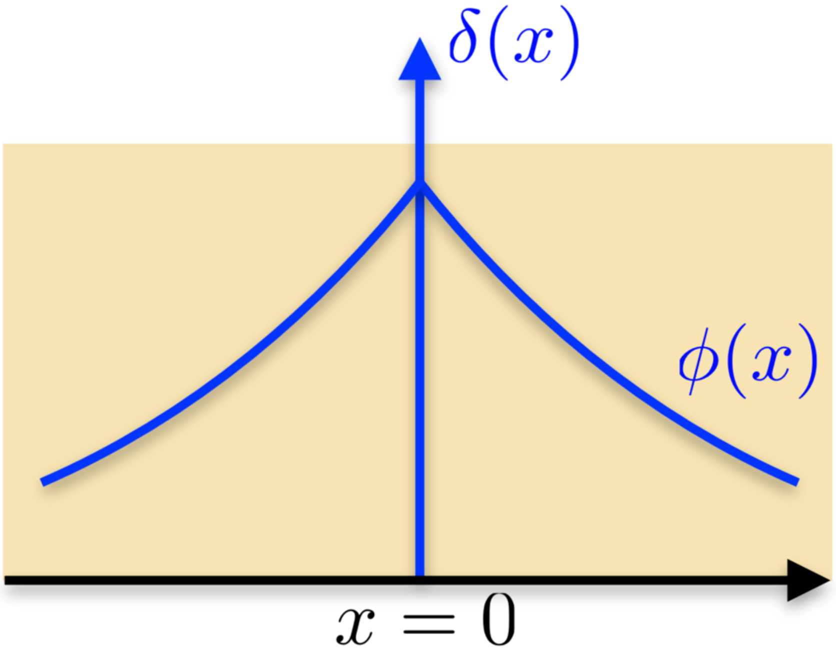
$$\text{Point Source: } \lim_{r \to 0} 4\pi r^2 J(r) = S$$

$$\text{Line Source: } \lim_{r \to 0} 2\pi r J(r) = S$$

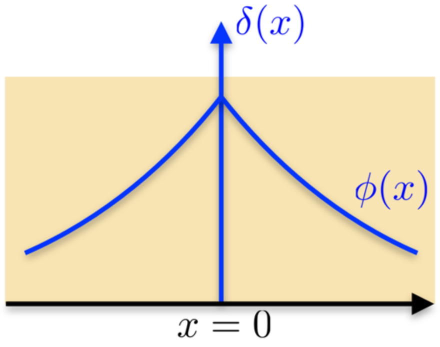
Practice

Infinite Planar Source

Consider an infinite planar source emitting S neutrons per cm^2/sec in an infinite diffusing medium. Determine $\phi(x)$.



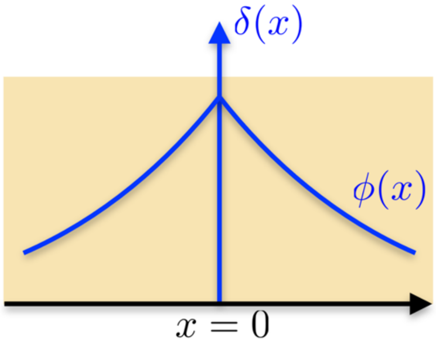
Infinite Planar Source



Consider an infinite planar source emitting S neutrons per cm^2/sec in an infinite diffusing medium. Determine $\phi(x)$.

$$\frac{1}{v}\frac{\partial\phi(\vec{r},t)}{\partial t} = S(\vec{r},t) - \Sigma_a(\vec{r})\phi(\vec{r},t) + \nabla\cdot D(\vec{r})\nabla\phi(\vec{r},t)$$

Infinite Planar Source



Consider an infinite planar source emitting S neutrons per cm^2/sec in an infinite diffusing medium. Determine $\phi(x)$.

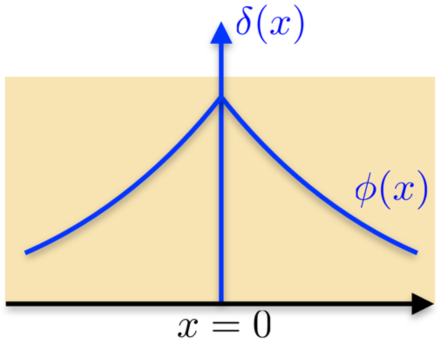
$$\frac{1}{v}\frac{\partial\phi(\vec{r},t)}{\partial t} = S(\vec{r},t) - \Sigma_a(\vec{r})\phi(\vec{r},t) + \nabla\cdot D(\vec{r})\nabla\phi(\vec{r},t)$$

Noting steady state, that source only exists at $x = 0$, and uniformity.

$$\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi = 0, \qquad x \neq 0$$

Berkeley UNIVERSITY OF CALIFORNIA

Infinite Planar Source



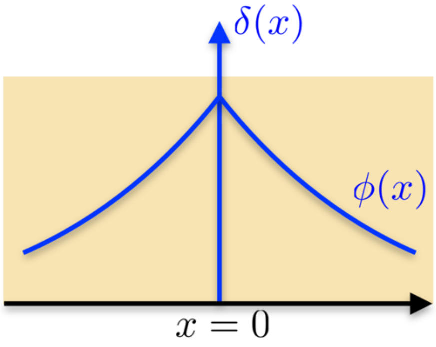
Consider an infinite planar source emitting S neutrons per cm^2/sec in an infinite diffusing medium. Determine $\phi(x)$.

$$\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi = 0, \qquad x \neq 0$$

The general solution to this second order differential eq is:

$$\phi = C_1 e^{-x/L} + C_2 e^{x/L}$$

Infinite Planar Source



Consider an infinite planar source emitting S neutrons per cm^2/sec in an infinite diffusing medium. Determine $\phi(x)$.

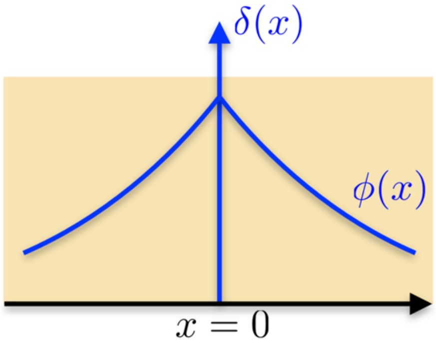
$$\phi = C_1 e^{-x/L} + C_2 e^{x/L}$$

BC 1) Flux must be finite, even as $x \to \infty$ or $x \to -\infty$

Considering when $x > 0$, constant C_2 must equal 0.

$$\phi(x) = C_2 e^{-x/L}, \qquad x > 0$$

Infinite Planar Source



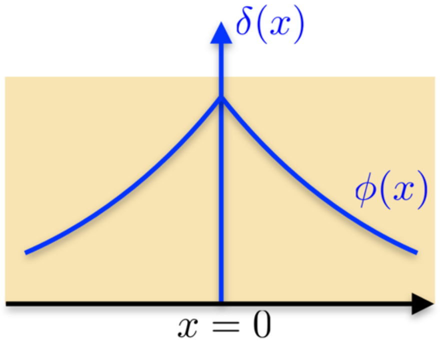
Consider an infinite planar source emitting S neutrons per cm^2/sec in an infinite diffusing medium. Determine $\phi(x)$.

$$\phi(x) = C_1 e^{-x/L}, \qquad x > 0$$

BC 2) Source at $x = 0$

$$\lim_{x \to 0} J(x) = \frac{S}{2}$$

Infinite Planar Source



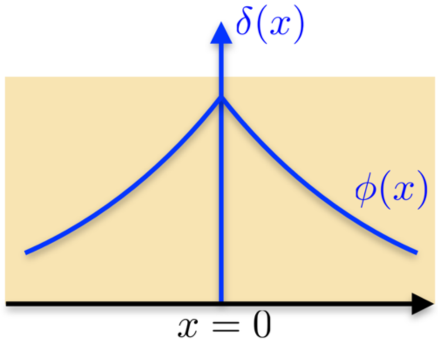
Consider an infinite planar source emitting S neutrons per cm^2/sec in an infinite diffusing medium. Determine $\phi(x)$.

$$\phi(x) = C_1 e^{-x/L}\,, \qquad \lim_{x \to 0} J(x) = \frac{S}{2}$$

Inserting this into Fick's Law

$$J = -D\frac{d\phi}{dx} = -D\frac{C_1}{L}e^{-x/L}$$

Infinite Planar Source



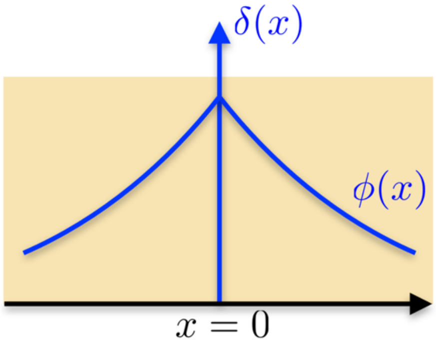
Consider an infinite planar source emitting S neutrons per cm^2/sec in an infinite diffusing medium. Determine $\phi(x)$.

$$\phi(x) = C_1 e^{-x/L}\,, \qquad \lim_{x\to 0} J(x) = \frac{S}{2}$$

Take the limit as $x \to 0$ to get:

$$C_1 = \frac{SL}{2D} \Rightarrow \phi(x) = \frac{SL}{2D} e^{-x/L}$$

Infinite Planar Source



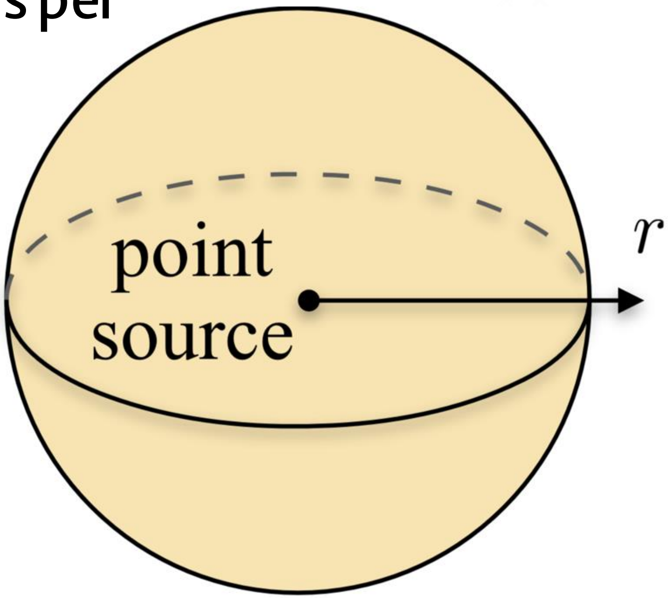
Consider an infinite planar source emitting S neutrons per cm^2/sec in an infinite diffusing medium. Determine $\phi(x)$.

You could solve this for $x < 0$, or just note the symmetry:

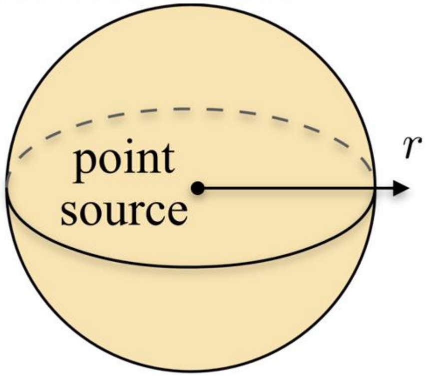
$$\phi(x) = \frac{SL}{2D} e^{-|x|/L}$$

Point Source

Consider a point source emitting S neutrons per second isotopically in an infinite diffusing medium. Determine $\phi(r)$.



Point Source



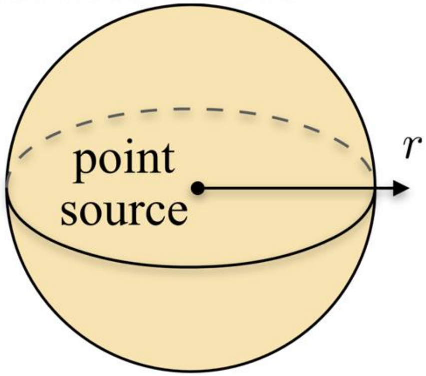
Consider a point source emitting S neutrons per second isotopically in an infinite diffusing medium. Determine $\phi(r)$.

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\phi}{dr}-\frac{\phi}{L^2}=0$$

The general solution to this second order differential eq is:

$$\phi = C_1\frac{e^{-r/L}}{r}+C_2\frac{e^{r/L}}{r}$$

Point Source



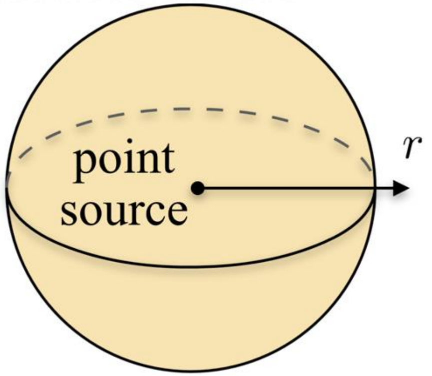
Consider a point source emitting S neutrons per second isotopically in an infinite diffusing medium. Determine $\phi(r)$.

$$\phi = C_1 \frac{e^{-r/L}}{r} + C_2 \frac{e^{r/L}}{r}$$

BC 1) Flux must be finite, even as $\mathrm{r} \to \infty$, so once again constant C_2 must equal 0

$$\phi = C_1 \frac{e^{-r/L}}{r} + 0$$

Point Source



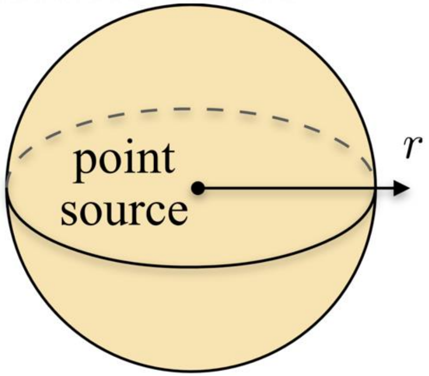
Consider a point source emitting S neutrons per second isotopically in an infinite diffusing medium. Determine $\phi(r)$.

$$\phi = C_1 \frac{e^{-r/L}}{r}$$

BC 2) Source at $\mathrm{r} = 0$

$$\lim_{r \to 0} 4\pi r^2 J(r) = S$$

Point Source



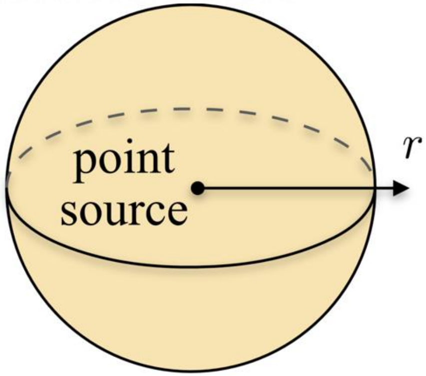
Consider a point source emitting S neutrons per second isotopically in an infinite diffusing medium. Determine $\phi(r)$.

$$\phi = C_1 \frac{e^{-r/L}}{r}, \qquad \lim_{r \to 0} 4\pi r^2 J(r) = S$$

Again, inserting this into Fick's Law:

$$J = -D\frac{d\phi}{dr} = DC_1\left(\frac{1}{rL} + \frac{1}{r^2}\right)e^{-\frac{r}{L}}$$

Point Source



Consider a point source emitting S neutrons per second isotopically in an infinite diffusing medium. Determine $\phi(r)$.

Using Fick's Law and the Source BC, we take the limit:

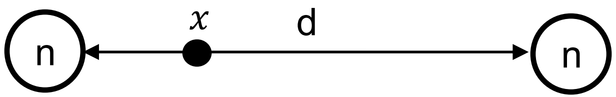
$$\lim_{r\to 0} r^2 J(r) = \frac{S}{4\pi} \Rightarrow C_1 = \frac{S}{4\pi D}$$

Finally we get:

$$\phi = \frac{S}{4\pi D}\frac{e^{-r/L}}{r}$$

Point Source Geometry

Say you have two point sources emitting S neutrons/sec are located 2a cm apart. Derive expressions for the flux at the point P_1 midway between the sources.



Point Source Geometry

Say you have two point sources emitting S neutrons/sec are located 2a cm apart. Derive expressions for the flux at the point P_1 midway between the sources.

$$\phi(P_1) = 2 \times \frac{Se^{-a/L}}{4\pi Da}$$