NE150/215M Introduction to Nuclear Reactor Theory
Spring 2022

Discussion 7: Diffusion Equation

March 30th, 2022

Helpful Readings: LE Ch.6, LB Ch.5

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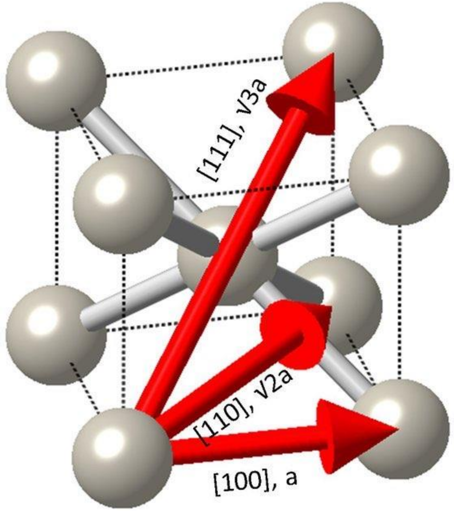
Homework Q/A

Problem 1) Assumptions of the Neutron Transport Equation

- We are checking the different situations and seeing if they violate any assumptions we've made
- Some of our assumptions technically aren't physically true but are still negligible
  - (i.e. Target nuclei aren't moving)
  - Don't overthink it!
- Reminder: A crystal's lattice looks like this:



Problem 2) Point Sources

- The diffusion equation doesn't apply since this problem is in a vacuum/void
  - There's no medium for the neutrons to diffuse through
  - This makes the problem easier!
- The key concept is understanding the difference between flux and current
  - Flux is scalar and thus additive, current is a vector – consider setting up the problem with vectors.

Problem 3) Two infinite planar sources

- Quite similar to problem 2, just using the diffusion equation
- Try to leverage symmetry
- Understand flux vs current

Problem 4) Thin absorbing sheet

- Assume that the sheet absorbs S' neutrons per unit area and unit time
  - Your answer will use this variable
- Since the entire medium is a source, your source term doesn't go away in your starting equation unlike our examples last discussion (see Equation 6.27 from Lewis)
- How can we change our boundary conditions to account for the absorption of neutrons rather than production?

Problem 5) Point Source in Moderator

a) A lot of math, sorry. Note your general solution:

$$\phi(r) = \frac{c_1 \cosh\left(\frac{r}{L}\right)}{r} + \frac{c_2 \sinh\left(\frac{r}{L}\right)}{r}$$

b) The leakage rate will be the current passing through the entire sphere's surface area.

Review

Non-Multiplying Media Problems

- What we've done so far
- No neutrons from fission
- Known neutron source

$$-D\frac{d^2\phi(x)}{dx^2} + \Sigma_a\phi(x) = S(\vec{r}, t)$$

Multiplying Medium Problems

- Neutrons come from fission
- No external source
- Homogenous equation

$$S(\vec{r},t) = \nu\Sigma_f(\vec{r})\phi(\vec{r},t)$$

The diffusion equation becomes:

$$-D\frac{d^2\phi(x)}{dx^2} + \Sigma_a\phi(x) = \nu\Sigma_f\phi(x)$$

Separation of Variables

- Assuming the shape of the flux largely depends on the geometry
- True when reactor conditions are relatively stable
  - Doesn't apply to rapid changes in reactor power

$$\phi(x,t) = \Phi(x)T(t)$$

Math Review: Eigenvalue Equations

- An eigenfunction of an operator *H* is a function *f* that satisfies the following, where *c* is an eigenvalue:

$$Hf = cf$$

For our application:

$$H = \nabla^2, \qquad f = \phi, \qquad c = B_n^2$$

$$\frac{d^2\phi}{dx^2} = -\frac{\bar{\nu}\Sigma_f - \Sigma_a + \frac{\lambda_a}{v}}{D}\phi(x) = B_n^2\phi(x)$$

Eigenvalue Problems

- Infinitely many functions satisfy this

$$\frac{d^2\phi_n}{dx_n^2} + B_n^2\phi_n(x) = 0, \qquad for\ n = 1,2,3\ldots$$

- We showed that we can ignore the higher order partial solutions because they go to zero very quickly
- We are looking for the **<u>fundamental</u>** mode

$$\phi(x,t) \xrightarrow[t\to\infty]{} A_1 e^{-\lambda_1 t}\cos(B_0 x)$$

Buckling and Criticality Condition

- Material buckling describes the neutron production and absorption of an infinite fuel material
- Geometric buckling describes the neutron leakage
- If the system is critical, it is in steady state: $\lambda_1 = 0$
- The criticality condition is such that they are equal:

$$B_m^2 = B_g^2 \quad \text{(for bare systems)}$$

Material Buckling

- Considers only the material properties in an infinite medium (recall the two factor formula)

$$k_\infty = \frac{\nu\Sigma_f}{\Sigma_a}$$

$$B_m^2 = \frac{k_\infty - 1}{L^2} = \frac{\nu\Sigma_f - \Sigma_a}{D}$$

Geometric Buckling and Flux Shapes

Geometry	Dimensions	Buckling, B_g^2	Flux Shape, ϕ
Infinite Slab	a	$\left(\frac{\pi}{\tilde{a}}\right)^2$	$A\cos\left(\frac{\pi}{\tilde{a}}x\right)$
Parallelepiped	a, b, c	$\left(\frac{\pi}{\tilde{a}}\right)^2+\left(\frac{\pi}{\tilde{b}}\right)^2+\left(\frac{\pi}{\tilde{c}}\right)^2$	$A\cos\left(\frac{\pi}{\tilde{a}}x\right)\cos\left(\frac{\pi}{\tilde{b}}y\right)\cos\left(\frac{\pi}{\tilde{c}}z\right)$
Infinite Cylinder	R	$\left(\frac{2.405}{\tilde{R}}\right)^2$	$AJ_0\left(\frac{2.405}{\tilde{R}}r\right)$
Finite Cylinder	R, H	$\left(\frac{2.405}{\tilde{R}}\right)^2+\left(\frac{\pi}{\tilde{H}}\right)^2$	$AJ_0\left(\frac{2.405}{\tilde{R}}r\right)\cos\left(\frac{\pi}{\tilde{H}}z\right)$
Sphere	R	$\left(\frac{\pi}{\tilde{R}}\right)^2$	$A\frac{1}{r}\sin\left(\frac{\pi}{\tilde{R}}r\right)$

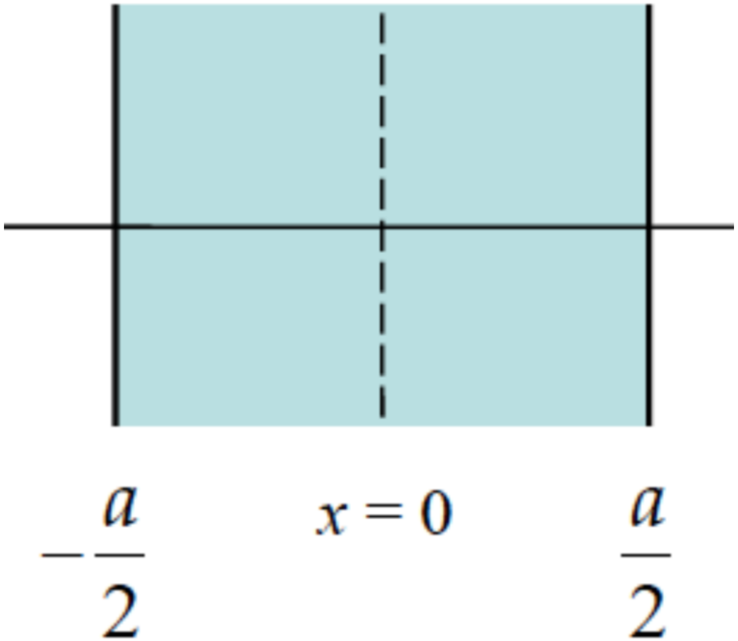
Other Diffusion Equation Tips

- Calculating the power can allow you to solve for your final constant once you have the flux shape
- You can also use separation of variables for different spatial dimensions $\phi(x, y, z) = X(x)Y(y)Z(z)$
- The dependence of B_m^2 on k_∞ allows you to use the two factor formula, which can include multiple isotopes

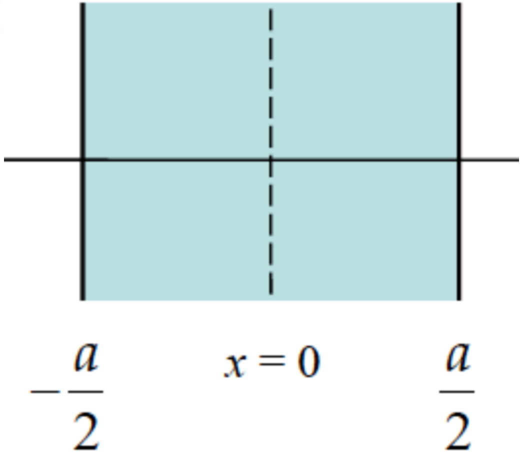
Practice

Critical Multiplying Bare Slab

Consider a bare slab made of a uniform neutron multiplying material that is critical. Determine $\phi(r)$.



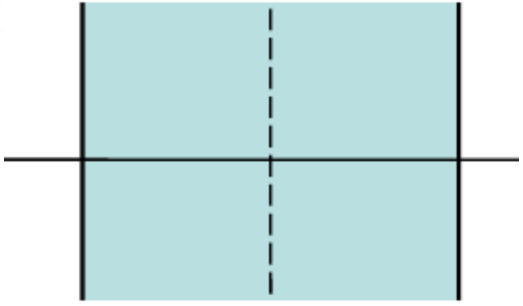
Critical Multiplying Bare Slab



Consider a bare slab made of a uniform neutron multiplying material that is critical. Determine $\phi(r)$.

$$\frac{1}{v}\frac{\partial\phi(\vec{r},t)}{\partial t} = S(\vec{r},t) - \Sigma_a(\vec{r})\phi(\vec{r},t) + \nabla\cdot D(\vec{r})\nabla\phi(\vec{r},t)$$

Critical Multiplying Bare Slab

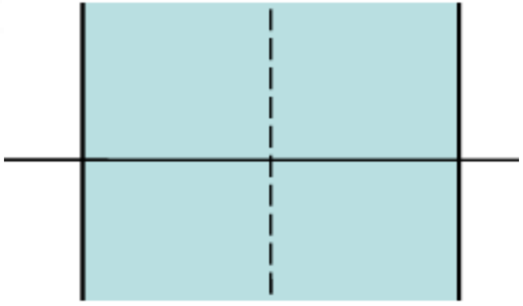


Consider a bare slab made of a uniform neutron multiplying material that is critical. Determine $\phi(r)$.

$$\frac{1}{v}\frac{\partial \phi(\vec{r},t)}{\partial t} = S(\vec{r},t) - \Sigma_a(\vec{r})\phi(\vec{r},t) + \nabla \cdot D(\vec{r})\nabla\phi(\vec{r},t)$$

- Critical -> Steady state $(d\phi/\partial t = 0)$
- Uniform (D, Σ_a constant)
- Multiplying $S(\vec{r},t) = \nu\Sigma_f\phi(\vec{r},t)$
- Slab (1D cartesian Laplacian)

Critical Multiplying Bare Slab



Consider a bare slab made of a uniform neutron multiplying material that is critical. Determine $\phi(r)$.

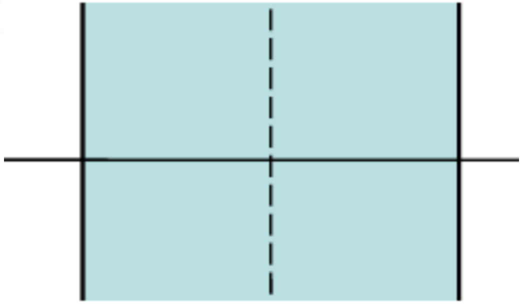
We end up with:

$$0 = \mu\Sigma_f\phi(x) - \Sigma_a\phi(x) + D\frac{d^2\phi}{dx^2}$$

Rearranging to get into eigenfunction format:

$$\frac{d^2\phi}{dx^2} + \frac{\bar{\nu}\Sigma_f - \Sigma_a}{D}\phi(x) = 0, \qquad \frac{\bar{\nu}\Sigma_f - \Sigma_a}{D} = B_m^2$$

Critical Multiplying Bare Slab



Consider a bare slab made of a uniform neutron multiplying material that is critical. Determine $\phi(r)$.

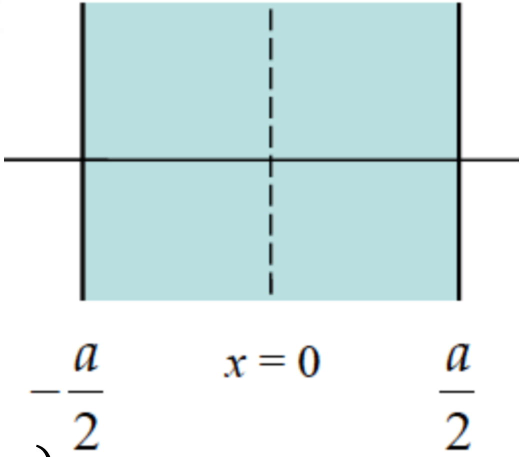
$$\frac{d^2\phi}{dx^2} + B_m^2(x) = 0$$

The general solution to this equation:

$$\phi(x) = C_1\cos(B_m x) + C_2\sin(B_m x)$$

Note: B_m is used instead of B_g when the reactor is critical

Critical Multiplying Bare Slab



Consider a bare slab made of a uniform neutron multiplying material that is critical. Determine $\phi(r)$.

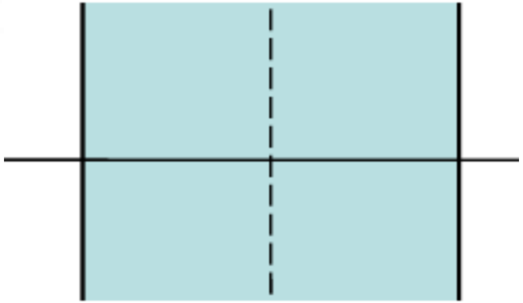
$$\phi(x) = C_1 \cos(B_m x) + C_2 \sin(B_m x)$$

BC 1) Symmetry

The sine term drops out since the flux solution has to be symmetric

$$C_2 = 0$$

Critical Multiplying Bare Slab



Consider a bare slab made of a uniform neutron multiplying material that is critical. Determine $\phi(r)$.

$$\phi(x) = C_1 \cos(B_m x)$$

BC 2) Vacuum

$$\phi\left(\pm\frac{\tilde{a}}{2}\right) = 0$$

$$\Rightarrow C_1 \cos\left(B_m \frac{\tilde{a}}{2}\right) = 0 \Rightarrow B_m \frac{\tilde{a}}{2} = \frac{\pi}{2}(1 + 2n), \qquad n = 0,1,2 \ldots$$

Higher Modes

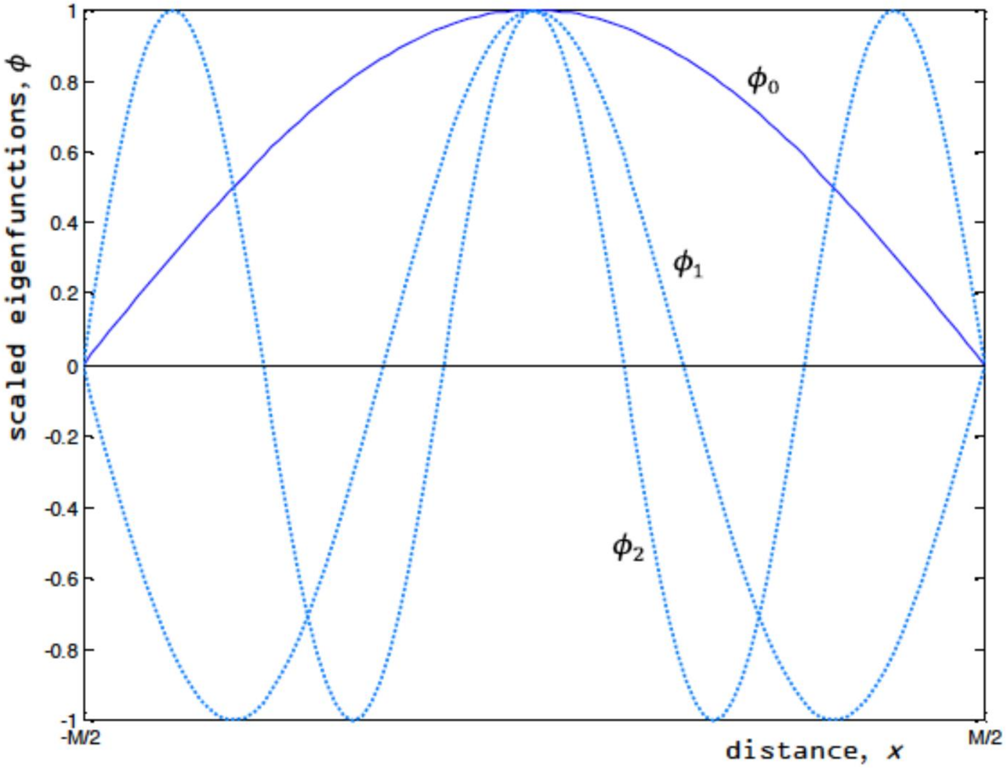
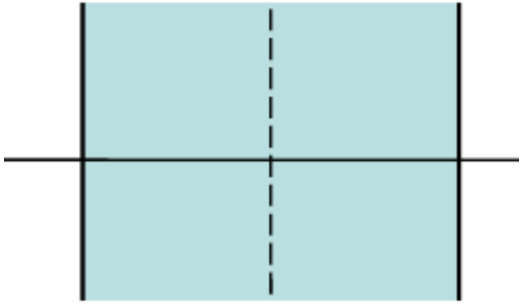


Figure 9. Eigenfunctions for the slab reactor in one-group model.

Critical Multiplying Bare Slab



Consider a bare slab made of a uniform neutron multiplying material that is critical. Determine $\phi(r)$.

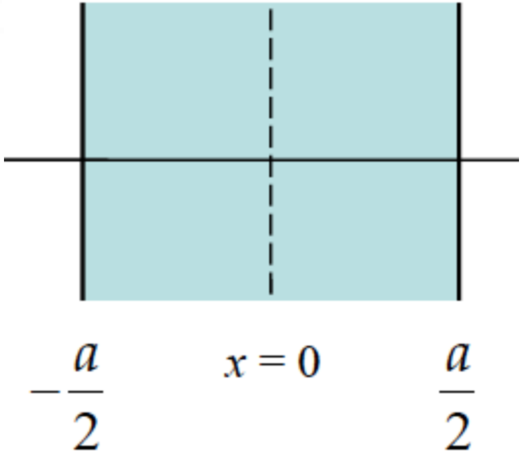
$$\phi(x) = C_1 \cos(B_m x)$$

BC 2) Vacuum

If $\mathrm{n} \geq 1$, the flux will be negative somewhere, so the solution we want is for $n = 0$.

$$B_m \frac{\tilde{a}}{2} = \frac{\pi}{2}(1 + 2n) \Rightarrow B_m \frac{\tilde{a}}{2} = \frac{\pi}{2} \Rightarrow B_m = \frac{\pi}{\tilde{a}}$$

Critical Multiplying Bare Slab

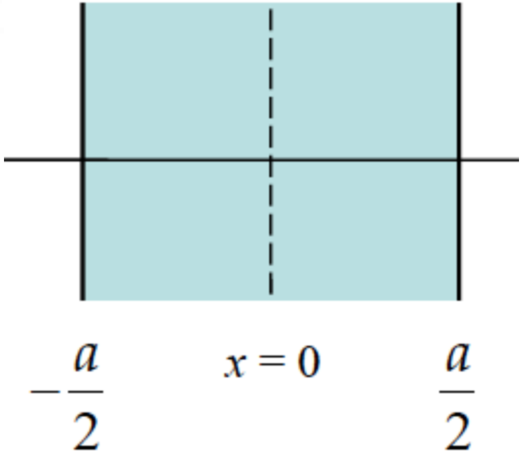


Consider a bare slab made of a uniform neutron multiplying material that is critical. Determine $\phi(r)$.

$$B_m^2 = \left(\frac{\pi}{\tilde{a}}\right)^2 = B_g^2$$

$$\phi(x) = C_1 \cos\left(\frac{\pi}{\tilde{a}} x\right)$$

Critical Multiplying Bare Slab



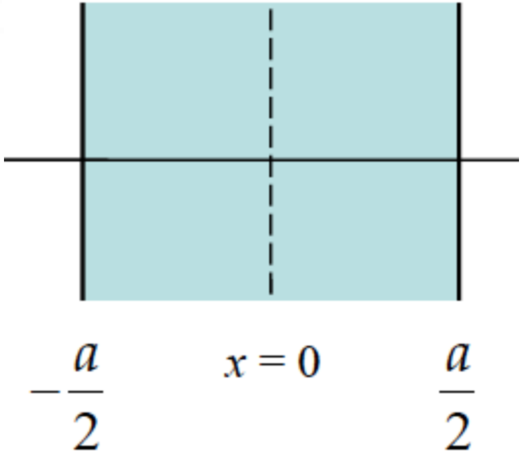
Consider a bare slab made of a uniform neutron multiplying material that is critical. Determine $\phi(r)$.

$$B_m^2 = \left(\frac{\pi}{\tilde{a}}\right)^2 = B_g^2$$

$$\phi(x) = C_1 \cos\left(\frac{\pi}{\tilde{a}} x\right)$$

We now have the shape of the flux! The magnitude, as given by C_1, can be calculated with the power. Power can be determined by integrating the flux. Suppose the slab has a power of $q''(W/cm^2)$

Critical Multiplying Bare Slab



Consider a bare slab made of a uniform neutron multiplying material that is critical. Determine $\phi(r)$.

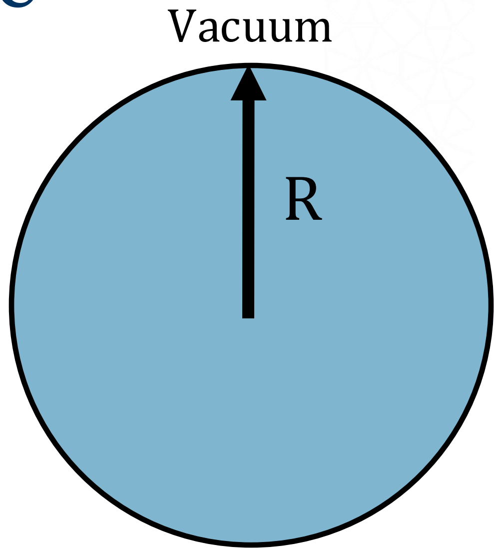
$$q'' = \int_{-a/2}^{a/2} E_f \phi(x) \Sigma_f \, dx = E_f \Sigma_f \int_{-a/2}^{a/2} C_1 \cos\left(\frac{\pi}{\tilde{a}} x\right) dx$$

$$\approx C_1 E_f \Sigma_f \frac{2a}{\pi}, \qquad (\tilde{a} = a)$$

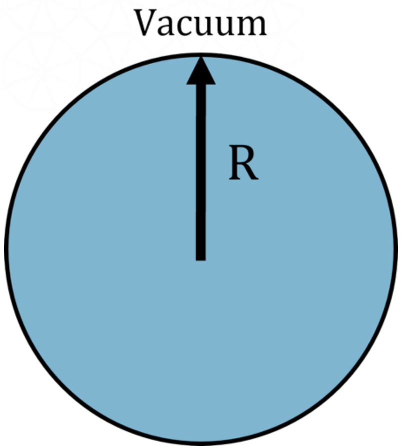
$$C_1 = \frac{q'' \pi}{E_f \Sigma_f 2a} \quad units \; \frac{W}{cm^2 \; J \; cm^{-1} \; cm} = \frac{W}{cm^2 s}$$

Critical Multiplying Bare Sphere

Consider a bare sphere made of a uniform neutron multiplying material that is critical. Determine $\phi(r)$.



Critical Multiplying Bare Sphere

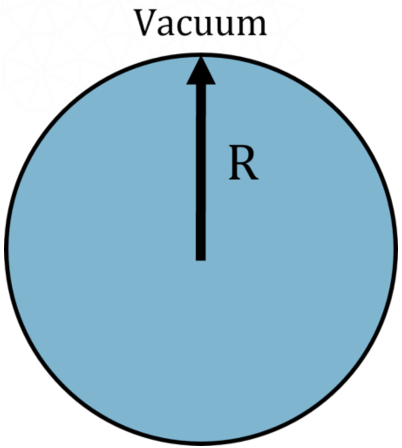


Consider a bare sphere made of a uniform neutron multiplying material that is critical. Determine $\phi(r)$.

Using the same set of simplifications as before:

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\phi}{dr} + B_m^2\phi(r) = 0$$

Critical Multiplying Bare Sphere



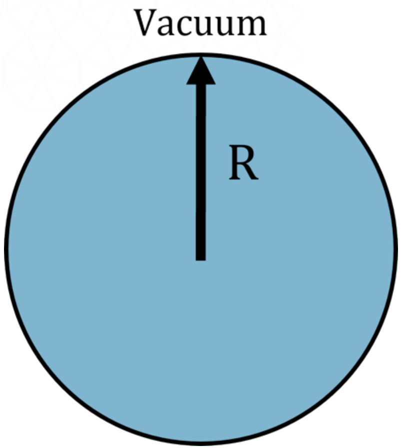
Consider a bare sphere made of a uniform neutron multiplying material that is critical. Determine $\phi(r)$.

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\phi}{dr} + B_m^2\phi(r) = 0$$

This equation has the general solution:

$$\phi(r) = C_1\frac{\cos(B_m r)}{r} + C_2\frac{\sin(B_m r)}{r}$$

Critical Multiplying Bare Sphere



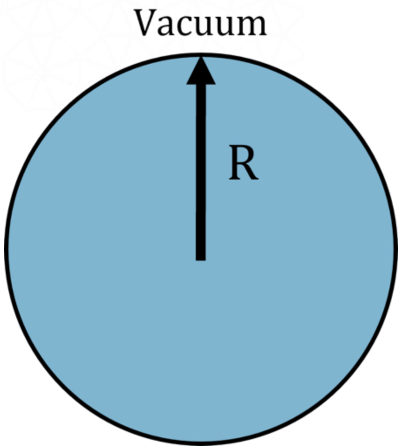
Consider a bare sphere made of a uniform neutron multiplying material that is critical. Determine $\phi(r)$.

$$\phi(r) = \cancel{C_1 \frac{\cos(B_m r)}{r}} + C_2 \frac{\sin(B_m r)}{r}$$

BC 1) Finite flux

$$\lim_{r \to 0} \phi(r) < \infty \quad \text{but} \quad \lim_{r \to 0} \frac{\cos(B_m r)}{r} \to \infty \quad \text{so} \quad C_1 = 0$$

Critical Multiplying Bare Sphere



Consider a bare sphere made of a uniform neutron multiplying material that is critical. Determine $\phi(r)$.

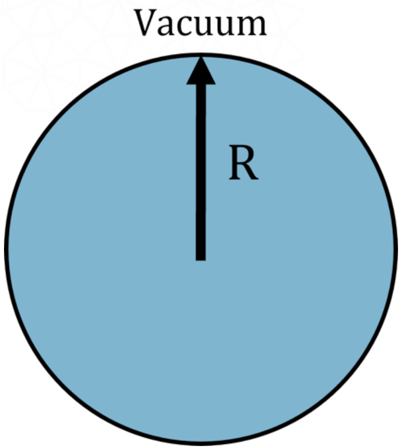
$$\phi(r) = C_2 \frac{\sin(B_m r)}{r}$$

BC 2) Vacuum Boundary

$$\phi(\tilde{R}) = 0 = C_2 \frac{\sin(B_m \tilde{R})}{\tilde{R}} \Rightarrow B_m \tilde{R} = n\pi, \qquad n = 0,1,2,\ldots$$

$n = 0$ is trivial, $\mathrm{n} > 1$ gives negative flux, so $\mathrm{n} = 1$ is real

Critical Multiplying Bare Sphere



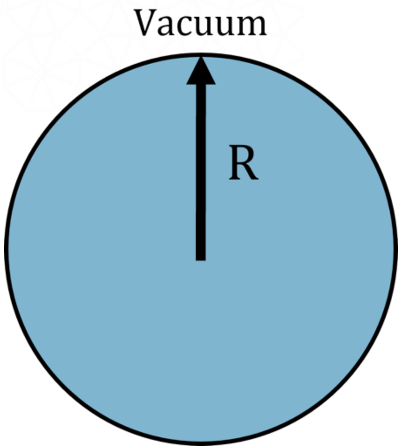
Consider a bare sphere made of a uniform neutron multiplying material that is critical. Determine $\phi(r)$.

$$B_m = \frac{\pi}{\tilde{R}}, \qquad B_m^2 = B_g^2 = \left(\frac{\pi}{\tilde{R}}\right)^2$$

Finally, our flux shape is

$$\phi(r) = C_2 \frac{1}{r} \sin\left(\frac{\pi}{\tilde{R}} r\right)$$

Critical Multiplying Bare Sphere



Consider a bare sphere made of a uniform neutron multiplying material that is critical. Determine $\phi(r)$.

$$\phi(r) = C_2 \frac{1}{r} \sin\left(\frac{\pi}{\tilde{R}} r\right)$$

Again, we find the using the power P produced in the sphere.

$$P = \int_0^R 4\pi r^2 dr E_f \Sigma_f \phi(r) = C_2 E_f \Sigma_f 4\pi \int_0^R r \sin\left(\frac{\pi}{\tilde{R}} r\right) dr$$

$$P \approx C_2 E_f \Sigma_f 4R^2 \quad (R \approx \tilde{R}) \Rightarrow C_2 = \frac{P}{E_f \Sigma_f 4R^2} \quad \frac{W}{J\ cm^{-1}\ cm^2} = \frac{W}{cm \cdot s}$$