NE150/215M Introduction to Nuclear Reactor Theory
Spring 2022

# Discussion 8: Diffusion Equation

April 6th, 2022

Helpful Readings: LE Ch.7, LB Ch.6

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# Warmup

1) Suppose the flux of a cylindrical reactor is given by:

$$\phi(r,z) = \frac{4\times 10^{12}}{\Sigma_f} J_0\left(\frac{2.4}{50}r\right)\cos\left(\frac{\pi}{120}z\right)\ \left[\frac{n}{cm^2\cdot s}\right]$$

What is the power density at r = 9cm, z = 34cm?

(Note: $J_0(0.432) \approx 0.969,\ 1\ \mathrm{Me}V = 1.602\times 10^{-13} J$)

2) A critical, bare cylindrical reactor has a radius of 160cm and a geometric buckling $B_m^2 = 4\times 10^{-4}\ cm^{-2}$. What is the height?

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$$p = E_f \Sigma_f \phi$$

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What is the power density at r = 9cm, z = 34cm?

(Note: $J_0(0.432) \approx 0.969$, $1\text{ MeV} = 1.602 \times 10^{-13} J$)

$$p = E_f \Sigma_f \phi = 200\, MeV(4 \times 10^{12}) J_0(0.432)\cos(0.890)$$

$$p = 78.15 \frac{W}{cm^3}$$

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$$B_m^2 = B_g^2 = \left(\frac{2.405}{\tilde{R}}\right)^2 + \left(\frac{\pi}{\tilde{H}}\right)^2$$

$$\tilde{H} = \frac{\pi}{\sqrt{B_m^2 - \left(\frac{2.405}{\tilde{R}}\right)^2}} = \frac{\pi}{\sqrt{4 \times 10^{-4} cm^{-2} - \left(\frac{2.405}{20cm}\right)^2}} = 238\text{cm}$$

# Homework Q/A

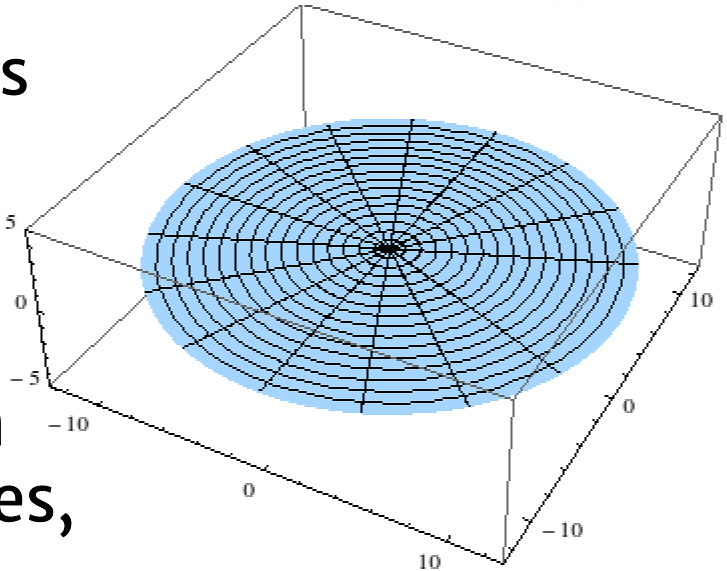
# Review

# Math Review: Bessel Functions

The solutions to the following differential equation are known as Bessel functions $J_n(x)$ and $Y_n(x)$

$$x^2 \frac{d^2 f}{dx^2} + x \frac{df}{dx} + (x^2 - n^2) f = 0$$

Describes things like vibrations in drums, dynamics of floating bodies, and flux in cylindrical reactors



# Math Review: Bessel Functions

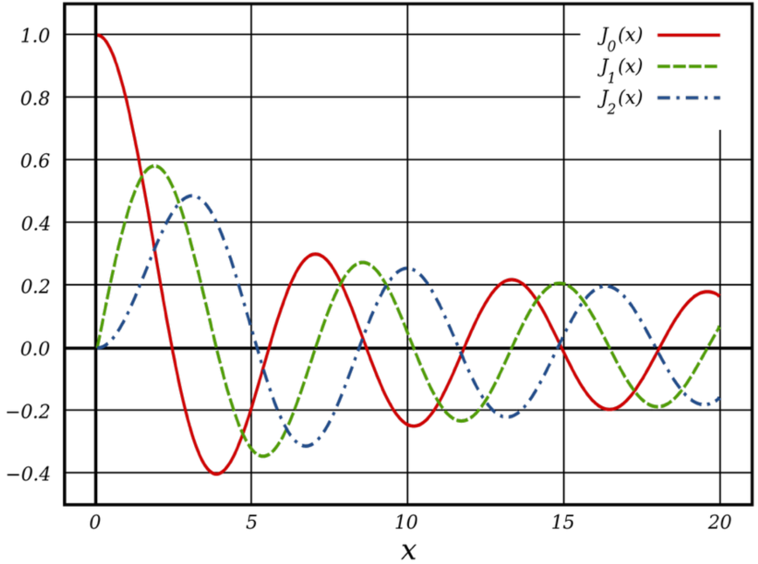
- With the diffusion equation, these usually appear in the general solution of cylindrical geometries
- They also have higher order mode solutions
  - We only care about $n = 0$

$$\frac{1}{r}\frac{d}{dr}r\frac{d\phi_n(r)}{dr} + B_n^2\phi_n(r) = 0$$

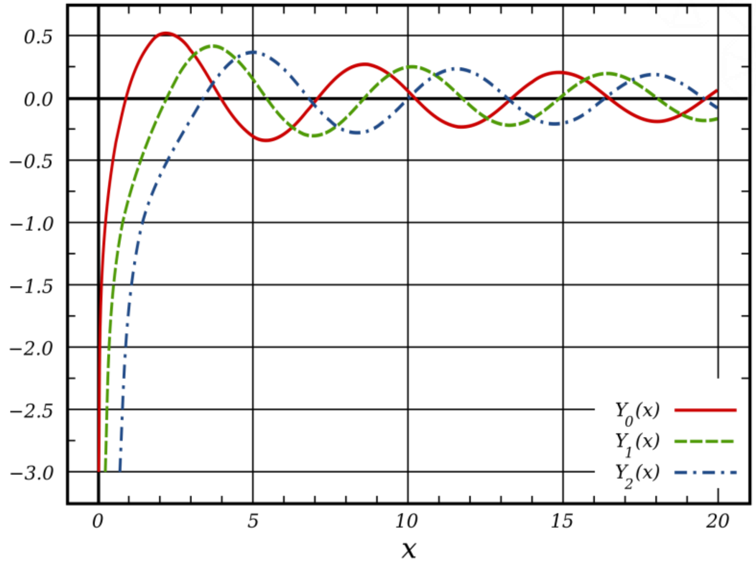
$$\Rightarrow \phi_n = C_1 J_0(B_n r) + C_2 Y_0(B_n r)$$

# Math Review: Bessel Functions

$J_n(x)$, First Kind




$Y_n(x)$, Second Kind



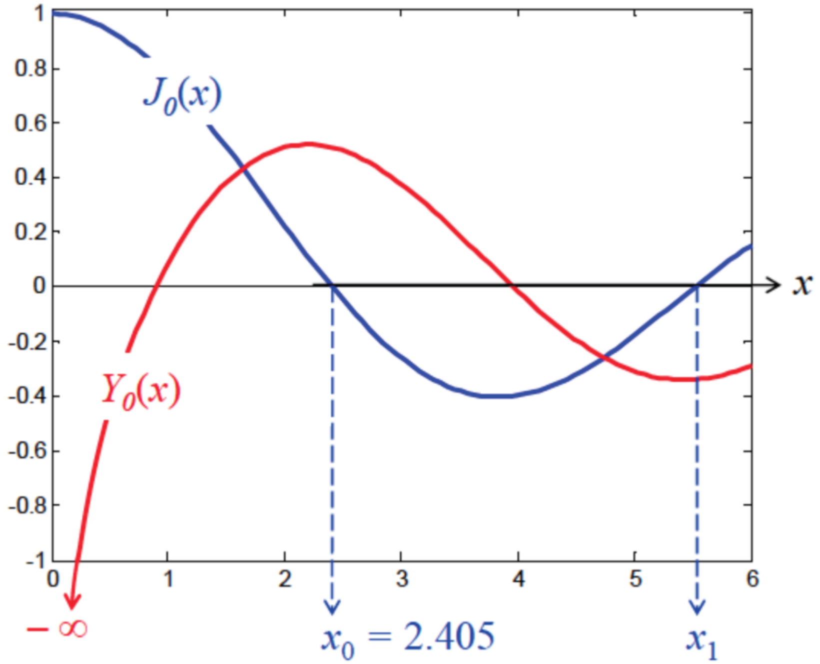

Notice where the functions are negative

# Math Review: Bessel Functions

- Evaluate numerically (Wolfram alpha)
- You may see 2.405 written as $\nu_0$

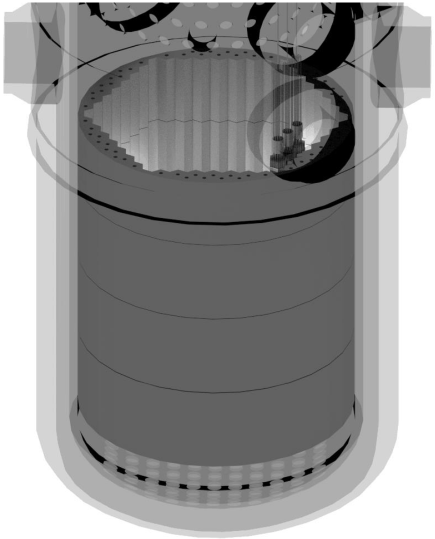
$$\frac{d}{dx}[x^m J_m(x)] = x^m J_{m-1}(x)$$

$$\int_0^u u' J_0(u') du' = u J_1(u)$$



# Reflectors

- A reflector is a non-multiplying layer around a reactor core that scatters neutrons back in
  - Reduces neutron leakage
- Materials like graphite, beryllium, steel
  - Low absorption
  - High elastic scattering
  - In thermal reactors, lower A desired for better moderation



# Reflector Effects

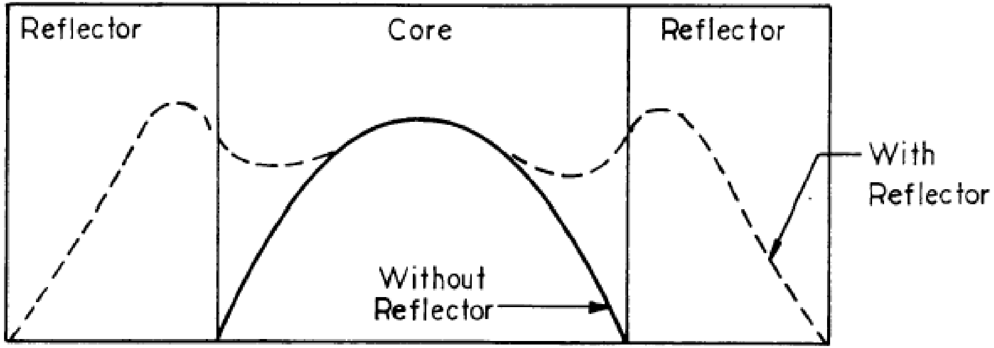
- Reduces critical size of active core (or enrichment)

$$Reflector\ Savings \equiv R_{critical}^{bare} - R_{critical}^{reflected}$$

- Reduces power peaking factor (or max-to-average flux ratio)

$$PPF \equiv \frac{\phi_{max}}{\phi_{avg}}, \qquad where\ \phi_{ave} = \frac{1}{V}\int_V \phi dV$$

- Increases neutron flux in core near the boundary



# Reflectors and the Diffusion Equation

- Reflectors create additional regions you must solve separately for
  - Interface boundary condition should be used
  - Reflectors aren't multiplying so no source
- $B_m^2 \neq B_G^2$ no longer applies, so finding the criticality condition takes extra work

$$ex\!: \frac{J_{core}(R)}{\phi_{core}(R)} = \frac{J_{reflector}(R)}{\phi_{reflector}(R)} \text{ (from interface)}$$

# Recall: Interface Boundary Condition

The flux and normal component of the current are continuous at an interface $\vec{r}_s$ between two different materials.

$$\phi_1(\vec{r}_s) = \phi_2(\vec{r}_s)$$

$$J_1(\vec{r}_s) = J_2(\vec{r}_s) \quad or \quad -D_1\nabla\phi_1(\vec{r}_s) = -D_2\nabla\phi_2(\vec{r}_s)$$

# Diffusion Equation Problem Strategy

1. Start with the general equation and simplify, applying correct Laplacian for geometry
2. Write the corresponding general solution for each region
3. Eliminate terms that become negative, infinite, or violate symmetry within your geometry
4. Apply boundary conditions
5. Determine critical condition (bare vs reflected)
6. Integrate power density to get magnitude of flux shape

# Homework 8 Tips

- If geometry isn't bare, you must determine flux and current for each region to get the criticality condition
  - They can be related with the interface BC
- For simplifying criticality conditions, reviewing hyperbolic trig identities will be helpful

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^2 + e^{-2}}{2}$$

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x}, \qquad \coth x = \frac{\cosh x}{\sinh x}$$

# Practice

# Spherical Fast Reactor Assembly

A bare spherical critical assembly is made by a mixture of $^{239}$Pu and Na. What is the radius given that $N_{Pu} = 3.95 \times 10^{21}\ at/cm^3$ and $N_{Na} = 2.39 \times 10^{22}\ \mathrm{at/cm^3}$?

# Spherical Fast Reactor Assembly

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Critical condition for bare spherical reactor:

$$B_m^2 = B_g^2$$

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Critical condition for bare spherical reactor:

$$B_m^2 = \frac{k_\infty - 1}{L^2} = \frac{\nu\Sigma_f - \Sigma_a}{D}$$

$$B_g^2 = \left(\frac{\pi}{\tilde{R}}\right)^2$$

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$$\left(\frac{\pi}{\tilde{R}}\right)^2 = \frac{\nu\Sigma_f - \Sigma_a}{D}, \qquad \nu^{Pu} = 3.0$$

| | Pu | Na |
|---|---|---|
| $\sigma_a$ | 2.11 b | 0.0008 b |
| $\sigma_f$ | 1.85 b | 0.0 b |
| $\sigma_{tr}$ | 6.8 b | 3.3 b |

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$$\left(\frac{\pi}{\tilde{R}}\right)^2 = \frac{\nu\Sigma_f - \Sigma_a}{D}, \qquad \nu^{Pu} = 3.0$$

$$\Sigma_f = N_{Pu}\sigma_f^{pu} = 0.0073\ cm^{-1}$$

$$\Sigma_a = N_{Pu}\sigma_a^{pu} + N_{Na}\sigma_a^{Na} = 0.0084\ cm^{-1}$$

$$\Sigma_{tr} = N_{pu}\sigma_{tr}^{Pu} + N_{Na}\Sigma_{tr}^{Na} = 0.1041\ cm^{-1}$$

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$$\left(\frac{\pi}{\tilde{R}}\right)^2 = \frac{\nu\Sigma_f - \Sigma_a}{D}, \qquad \nu^{Pu} = 3.0$$

$$\Sigma_f = 0.0073\ cm^{-1}, \qquad \Sigma_a = 0.0084\ cm^{-1}, \qquad \Sigma_{tr} = 0.1041\ cm^{-1}$$

$$D = \frac{1}{3\Sigma_{tr}} = 3.2027\ cm$$

# Spherical Fast Reactor Assembly

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$$\left(\frac{\pi}{\tilde{R}}\right)^2 = \frac{\nu\Sigma_f - \Sigma_a}{D}$$

$$B_m^2 = \frac{\left(3.0\,\frac{n}{fis}\right)(0.0073\ cm^{-1}) - (0.0084 cm^{-1})}{3.2027\ cm}$$

$$B_m^2 = 0.0042\ cm^{-2}$$

# Spherical Fast Reactor Assembly

A bare spherical critical assembly is made by a mixture of $^{239}$Pu and Na. What is the radius given that $N_{Pu} = 3.95 \times 10^{21}\ at/cm^3$ and $N_{Na} = 2.39 \times 10^{22}\ \mathrm{at/cm^3}$?

$$\left(\frac{\pi}{\tilde{R}}\right)^2 = B_m^2 \Rightarrow \tilde{R} = \frac{\pi}{B_m} = \frac{\pi}{\sqrt{0.0042\ cm^{-2}}}$$

$$\tilde{R} = 48.5\ cm$$

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Remember that $\tilde{R}$ is the extrapolated radius

$$R = \tilde{R} - 2D = 48.5\ cm\ - 2 * 3.2027cm$$

$$R = 42.1cm$$

This isn't entirely negligible for this reactor!

# Spherical Fast Reactor Assembly

What is the probability a fission neutron is absorbed in this assembly?

$$D = 3.2027\ cm, \qquad \Sigma_a = 0.0084\ cm^{-1}, \qquad B_m^2 = 0.0042\ cm^{-2}$$

# Spherical Fast Reactor Assembly

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A neutron is either absorbed in the assembly, or it leaks out. We can calculate the leakage probability.

$$P_{absorbed} = 1 - P_{leakage} = P_{NL} = \frac{1}{1 + L^2 B^2}$$

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$$L^2 = \frac{D}{\Sigma_a} = \frac{3.2027\ cm}{0.0084\ cm^{-1}} = 381.27\ cm^2$$

# Spherical Fast Reactor Assembly

What is the probability a fission neutron is absorbed in this assembly?

$$D = 3.2027\ cm, \qquad \Sigma_a = 0.0084\ cm^{-1}, \qquad B_m^2 = 0.0042\ cm^{-2}$$

$$L^2 = 381.27\ cm^2$$

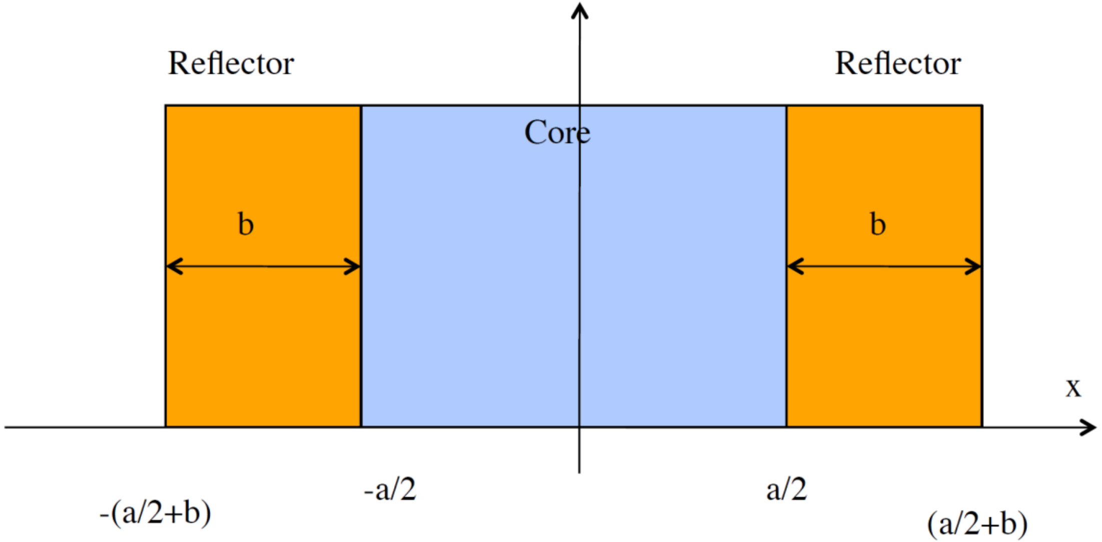
$$P_{NL} = \frac{1}{1 + L^2 B^2} = \frac{1}{1 + 381.27(0.0042\ cm^{-2})} = 38.44\%\ \text{absorbed}$$

$$\Rightarrow P_{leakage} = 1 - P_{absorbed} = 61.56\%\ \text{leaked}$$

A reflector could be helpful.

# Reflected Critical Slab Criticality Condition

Determine the criticality condition for a reflected critical slab.



# Reflected Critical Slab Criticality Condition

In lecture, we said the criticality condition was:

$$B_m D_c \tan\left(B_m^2 \frac{a}{2}\right) = \frac{D_r}{L_r} \coth\left(\frac{b}{L^2}\right)$$

But how did we get it? Let's take a closer look, since you need to do it for the homework.

# Reflected Critical Slab Criticality Condition

The flux in the core can be shown to have the following shape after using symmetry:

$$\phi_c(x) = C_c \cos(B_m x)$$

The flux in the reflector region can be shown to have following shape using vacuum boundary conditions:

$$\phi_r(x) = C_r \sinh\left(\frac{\frac{a}{2} + \tilde{b} - x}{L_r}\right)$$

# Reflected Critical Slab Criticality Condition

We then use the interface boundary condition:

$$\phi_{core}\left(\frac{a}{2}\right) = \phi_{ref}\left(\frac{a}{2}\right) \qquad J_{core}\left(\frac{a}{2}\right) = J_{ref}\left(\frac{a}{2}\right)$$

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$$C_c \cos\left(B_m \frac{a}{2}\right) = C_r \sinh\left(\frac{\frac{a}{2} + \tilde{b} - \frac{1}{2}}{L_r}\right)$$

$$C_c \cos\left(B_m \frac{a}{2}\right) = C_r \sinh\left(\frac{\tilde{b}}{L_r}\right)$$

# Reflected Critical Slab Criticality Condition

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$$C_c \cos\left(B_m \frac{a}{2}\right) = C_r \sinh\left(\frac{\tilde{b}}{L_r}\right)$$

$$J_{core}\left(\frac{a}{2}\right) = J_{ref}\left(\frac{a}{2}\right)$$

$$-D_c \frac{d\phi_c(x)}{dx}\bigg|_{x=\frac{a}{2}} = -D_r \frac{d\phi_r(x)}{dx}\bigg|_{x=\frac{a}{2}}$$

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$$-D_c \frac{d\phi_c(x)}{dx}\bigg|_{x=\frac{a}{2}} = -D_r \frac{d\phi_r(x)}{dx}\bigg|_{x=\frac{a}{2}}$$

$$D_c C_c B_m \sin\left(\frac{B_m a}{2}\right) = D_r C_r \frac{1}{L_r} \cosh\left(\frac{\tilde{b}}{L_r}\right)$$

# Reflected Critical Slab Criticality Condition

We can eliminate the constants and get our criticality condition by dividing our current equation with our flux equation:

$$\frac{J_{core}\left(\frac{a}{2}\right)}{\phi_{core}\left(\frac{a}{2}\right)} = \frac{J_{ref}\left(\frac{a}{2}\right)}{\phi_{ref}\left(\frac{a}{2}\right)}$$

$$\frac{D_c C_c B_m \sin\left(\frac{B_m a}{2}\right)}{C_c \cos\left(B_m \frac{a}{2}\right)} = \frac{D_r C_r \frac{1}{L_r} \cosh\left(\frac{\tilde{b}}{L_r}\right)}{C_r \sinh\left(\frac{\tilde{b}}{L_r}\right)}$$

# Reflected Critical Slab Criticality Condition

$$\frac{D_c C_c B_m \sin\left(\frac{B_m a}{2}\right)}{C_c \cos\left(B_m \frac{a}{2}\right)} = \frac{D_r C_r \frac{1}{L_r} \cosh\left(\frac{\tilde{b}}{L_r}\right)}{C_r \sinh\left(\frac{\tilde{b}}{L_r}\right)}$$

Simplifying, we get the criticality condition shown in class:

$$B_m D_c \tan\left(B_m \frac{a}{2}\right) = \frac{D_r}{L_r} \coth\left(\frac{\tilde{b}}{L_r}\right)$$