NE150/215M Introduction to Nuclear Reactor Theory
Spring 2022

Discussion 9: Multigroup Diffusion

April 13th, 2022

Helpful Readings: LB Ch. 5, 6

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Warmup

Suppose you have a bare cubical reactor with length a producing power P whose flux is given by:

$$\phi(x,y,z) = A\cos\left(\frac{\pi x}{\tilde{a}}\right)\cos\left(\frac{\pi y}{\tilde{b}}\right)\cos\left(\frac{\pi z}{\tilde{c}}\right)$$

Write an integral that allows you to solve for A.

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$$P = \int_V E_f \Sigma_f \phi(\vec{r})\, dV$$

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Write an integral that allows you to solve for A.

$$P = \int_V E_f \Sigma_f \phi(\vec{r})dV$$

$$P = \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{a}{2}}^{\frac{a}{2}} dy \int_{-\frac{a}{2}}^{\frac{a}{2}} dz\, E_f \Sigma_f \phi(x,y,z)$$

Warmup

Suppose you have a bare cubical reactor with length a producing power P whose flux is given by:

$$\phi(x,y,z) = A\cos\left(\frac{\pi x}{\tilde{a}}\right)\cos\left(\frac{\pi y}{\tilde{b}}\right)\cos\left(\frac{\pi z}{\tilde{c}}\right)$$

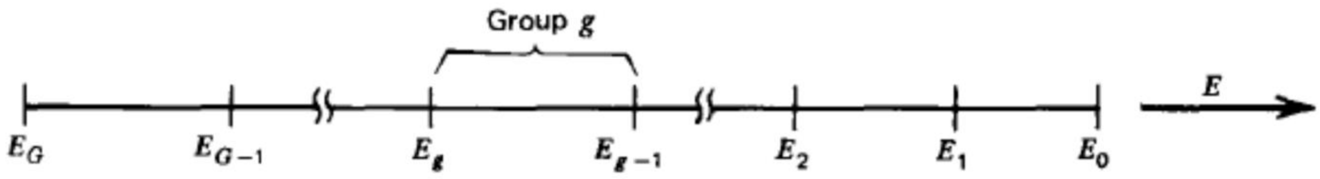
Write an integral that allows you to solve for A.

$$P = E_f \Sigma_f A \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{a}{2}}^{\frac{a}{2}} dy \int_{-\frac{a}{2}}^{\frac{a}{2}} dz \cos\left(\frac{\pi x}{\tilde{a}}\right)\cos\left(\frac{\pi y}{\tilde{b}}\right)\cos\left(\frac{\pi z}{\tilde{c}}\right)$$

Review

Energy Equation Dependent Diffusion

- So far, we've considered neutrons at one energy
- In reality, neutrons are born fast and scatter down to thermal energies, with many neutrons in between
- If we split the energy spectrum in multiple groups, we can get a better model of flux.



Multigroup Diffusion Equation

$$\frac{1}{v_g}\frac{\partial \phi_g}{\partial t} = \chi_g \sum_{g'=1}^{G} \nu_{g'} \Sigma_{f,g'} \phi_{g'} + \sum_{g'=1}^{G} \Sigma_{s,g' \to g} \phi_{g'} - \Sigma_{tot,g} \phi_g + D_g \nabla^2 \phi_g$$

Constants

$\Sigma_{f,g}$ $\Sigma_{tot,g}$ $\equiv$ Fission, total cross section for group g

$\Sigma_{s,g' \to g}$ $\equiv$ Scattering cross section from group g to g′

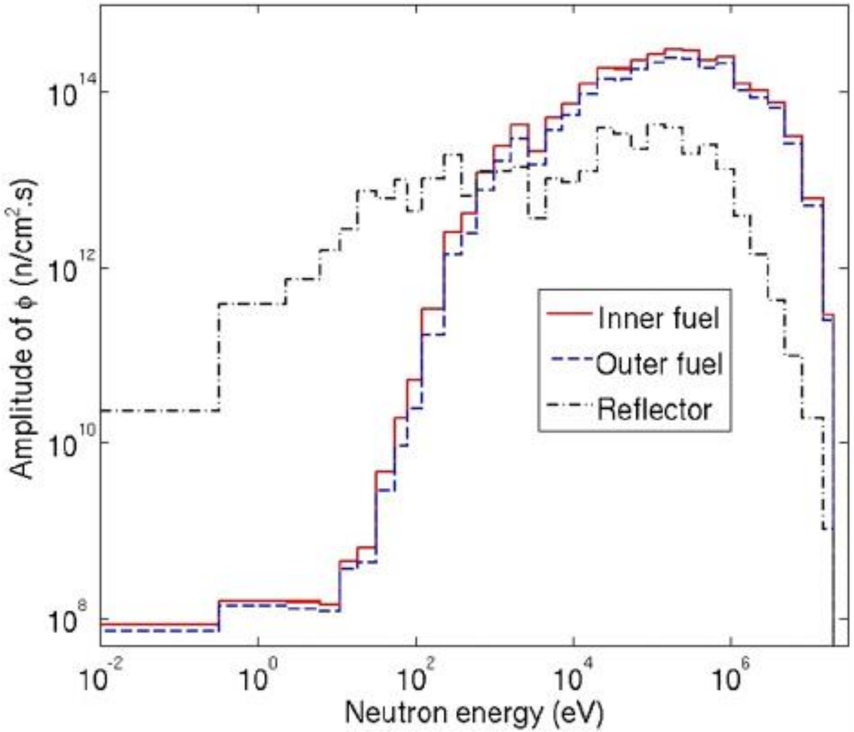
ν_g $\equiv$ Neutrons produced per fission for group g

χ_g $\equiv$ The fraction of fission neutrons emitted in group g

D_g $\equiv$ Diffusion coefficient for group g

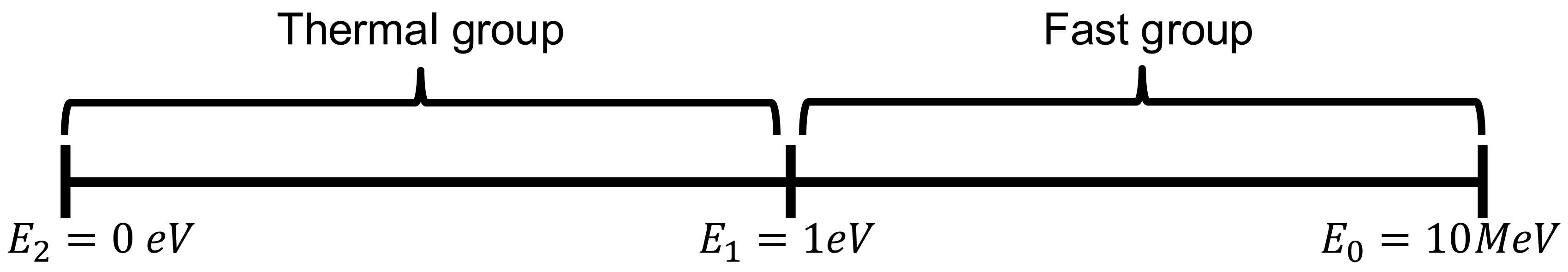
Application

- More accurate results
  - Using more appropriate data for different energies
- Understand impact of moderation better
- Better quantify damage to materials
  - Fast flux –> more DPA



Two Group Diffusion

- We're only considering fast and thermal neutrons
- All fission neutrons are born into the fast group
- There's no up-scattering from thermal group to fast group



Two Group Diffusion

Starting from the most general form...

$$-D_1\nabla^2\phi_1 + \Sigma_{t,1}\phi_1 = \sum_{g'=1}^{2}\Sigma_{s,g'\to 1}\phi_{g'} + \frac{1}{k}\chi_1\sum_{g'=1}^{2}\nu\Sigma_{f,g'}\phi_{g'}$$

$$-D_2\nabla^2\phi_2 + \Sigma_{t,2}\phi_2 = \sum_{g'=1}^{2}\Sigma_{s,g'\to 2}\phi_{g'} + \frac{1}{k}\chi_2\sum_{g'=1}^{2}\nu\Sigma_{f,g'}\phi_{g'}$$

Two Group Diffusion

Assumption: All fission neutrons are born into the fast group

$$(\chi_1 = 1, \qquad \chi_2 = 0)$$

$$-D_1\nabla^2\phi_1 + \Sigma_{t,1}\phi_1 = \sum_{g'=1}^{2}\Sigma_{s,g'\to 1}\phi_{g'} + \frac{1}{k}(1)\sum_{g'=1}^{2}\nu\Sigma_{f,g'}\phi_{g'}$$

$$-D_2\nabla^2\phi_2 + \Sigma_{t,2}\phi_2 = \sum_{g'=1}^{2}\Sigma_{s,g'\to 2}\phi_{g'} + \cancel{\frac{1}{k}(0)\sum_{g'=1}^{2}\nu\Sigma_{f,g'}\phi_{g'}}$$

Two Group Diffusion

Assumption: All fission neutrons are born into the fast group

$$(\chi_1 = 1, \qquad \chi_2 = 0)$$

$$-D_1 \nabla^2 \phi_1 + \Sigma_{t,1}\phi_1 = \sum_{g'=1}^{2} \Sigma_{s,g'\to 1}\phi_{g'} + \frac{1}{k}\sum_{g'=1}^{2} \nu\Sigma_{f,g'}\phi_{g'}$$

$$-D_2 \nabla^2 \phi_2 + \Sigma_{t,2}\phi_2 = \sum_{g'=1}^{2} \Sigma_{s,g'\to 2}\phi_{g'}$$

Two Group Diffusion

Assumption: There's no upscattering from thermal group to fast group

$$-D_1\nabla^2\phi_1 + \Sigma_{t,1}\phi_1 = \sum_{g'=1}^{2} \Sigma_{s,g'\to 1}\phi_{g'} + \frac{1}{k}\sum_{g'=1}^{2} \nu\Sigma_{f,g'}\phi_{g'}$$

$$-D_2\nabla^2\phi_2 + \Sigma_{t,2}\phi_2 = \sum_{g'=1}^{2} \Sigma_{s,g'\to 2}\phi_{g'}$$

Two Group Diffusion

Assumption: There's no upscattering from thermal group to fast group

$$-D_1\nabla^2\phi_1 + \Sigma_{t,1}\phi_1 = \Sigma_{s,1\to 1}\phi_1 + \frac{1}{k}\sum_{g'=1}^{2}\nu\Sigma_{f,g'}\phi_{g'}$$

$$-D_2\nabla^2\phi_2 + \Sigma_{t,2}\phi_2 = \Sigma_{s,1\to 2}\phi_1 + \Sigma_{s,2\to 2}\phi_2$$

Two Group Diffusion

We define a removal cross section:

$$\Sigma_{R,1} = \Sigma_{t,1} - \Sigma_{s,1\to 1}$$

The probability of neutrons within group 1 being removed as a result of scattering

$$-D_1\nabla^2\phi_1 + \Sigma_{R,1}\phi_1 = \frac{1}{k}\sum_{g'=1}^{2}\nu\Sigma_{f,g'}\phi_{g'}$$

$$-D_2\nabla^2\phi_2 + \Sigma_{a,2}\phi_2 = \Sigma_{s,1\to 2}\phi_1$$

Since there's no upscattering, $\Sigma_{R,2} = \Sigma_{t,2} - \Sigma_{s,2\to 2} = \Sigma_{\mathrm{a},2}$

Two Group Diffusion Derivation

When we expand the fission sum, we get our result from lecture:

$$-D_1\nabla^2\phi_1 + \Sigma_{R,1}\phi_1 = \frac{1}{k}\left(\nu_1\Sigma_{f,1}\phi_1 + \nu_2\Sigma_{f,2}\phi_2\right)$$

$$-D_2\nabla^2\phi_2 + \Sigma_{a,2}\phi_2 = \Sigma_{s,1\rightarrow 2}\phi_1$$

Two Group Diffusion

For uniform systems, you can substitute:

$$\nabla^2 \phi_g = -B^2 \phi_g$$

To get

$$D_1 B^2 \phi_1 + \Sigma_{R,1} \phi_1 = \frac{1}{k}\left(\nu_1 \Sigma_{f,1} \phi_1 + \nu_2 \Sigma_{f,2} \phi_2\right)$$
$$D_2 B^2 \phi_2 + \Sigma_{a,2} \phi_2 = \Sigma_{s,1\rightarrow 2} \phi_1$$

Two Group Diffusion

To solve for k, you can solve the system of equations:

$$\phi_2 = \frac{\Sigma_{s,1\to 2}}{B^2 D_2 + \Sigma_{a,2}} \phi_1$$

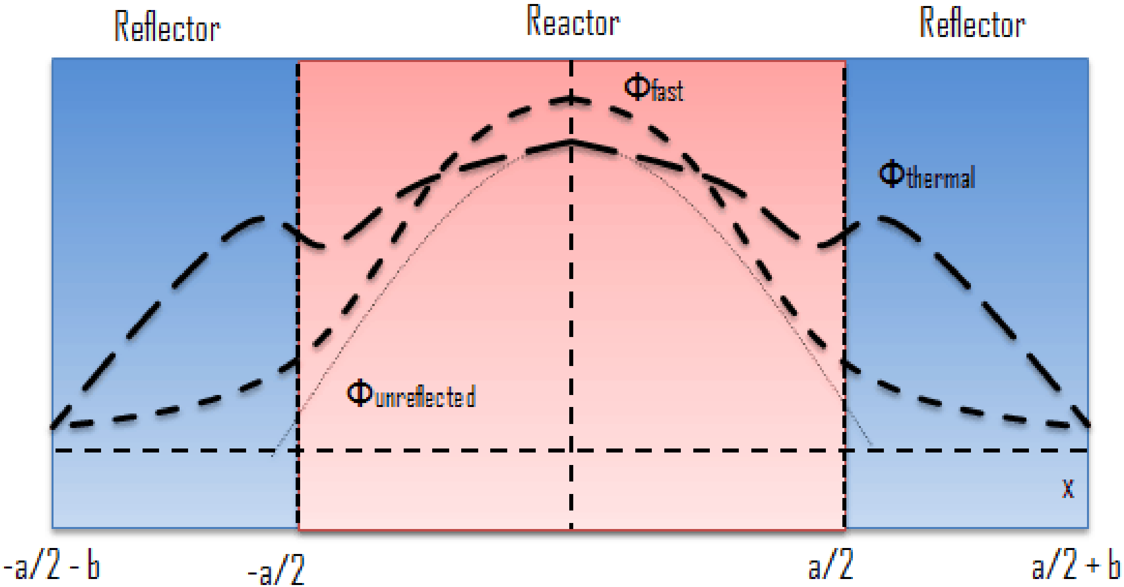
Substitute in ϕ_2

$$D_1 B^2 \phi_1 + \Sigma_{R,1} \phi_1 = \frac{1}{k}\left(\nu_1 \Sigma_{f,1} \phi_1 + \nu_2 \Sigma_{f,2} \frac{\Sigma_{s,1\to 2}}{B^2 D_2 + \Sigma_{a,2}} \phi_1\right)$$

Cancel out ϕ_1, and solve for k:

$$k = \frac{\nu_1 \Sigma_{f,1}}{D_1 B^2 + \Sigma_{R,1}} + \frac{\Sigma_{s,1\to 2} \nu_2 \Sigma_{f,2}}{(D_1 B^2 + \Sigma_{R,1})(D_2 B^2 + \Sigma_{a,2})}$$

Two Group Diffusion Solution on Slab with Reflector



http://www.nuclear-power.net/wp-content/uploads/2016/10/Two-Group-Method-Reflected-Reactor.png?a34b7f

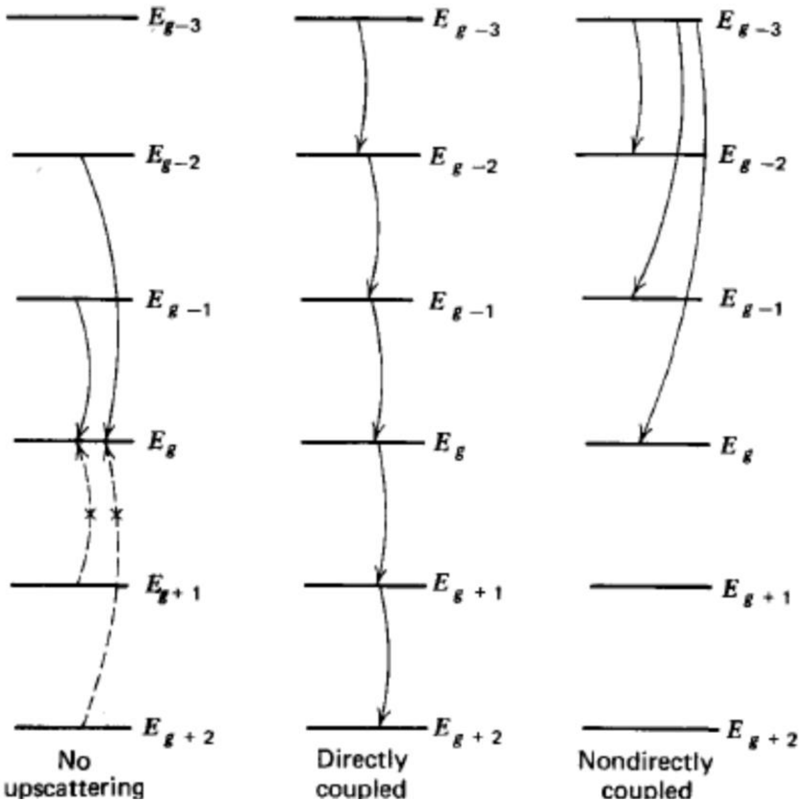
Multigroup Matrix Form

- For more than 2 groups, it's helpful to write the multigroup equation in matrix form
- M captures your diffusion, scattering, and removal terms
- F captures your fission terms

$$\underline{\underline{M}}\,\underline{\phi} = \frac{1}{k}\underline{\underline{F}}\,\underline{\phi}$$

Multigroup Coupling

- You may be given assumptions about which groups can scatter to which groups
- Usually, thermal neutrons can upscatter to other thermal groups, but fast neutrons can't



Example 1: Four-group Diffusion Equation

Suppose you have a slab reactor with uniform material properties. Derive the four-group diffusion equation in matrix form assuming

- Thermal neutrons are contained only in the lowest group.
- Fission neutrons are all born within the upper two groups.
- There is a direct coupling between all groups.
- There is no up-scattering.

Example 1: Four-group Diffusion Equations

Because there is no up–scattering, and there is direct coupling, all scattering leads to the group below

Group g	Thermal?	Fission Source?	Scattering to other groups
1	No	Yes	Down to 2
2	No	Yes	Down to 3
3	No	No	Down to 4
4	Yes	No	None

Example 1: Four-group Diffusion Equations

Group 1: $-D_1 \frac{d^2}{dx^2}\phi_1(x) + \Sigma_{R,1}\phi_1(x) = \frac{\chi_1}{k}\sum_{g'=1}^{4} \nu\Sigma_{f,g'}\phi_{g'}$

Group 2: $-D_2 \frac{d^2}{dx^2}\phi_2(x) + \Sigma_{R,2}\phi_2(x) - \Sigma_{S,1\rightarrow 2}\phi_1(x) = \frac{\chi_2}{k}\sum_{g'=1}^{4} \nu\Sigma_{f,g'}\phi_{g'}$

Group 3: $-D_3 \frac{d^2}{dx^2}\phi_3(x) + \Sigma_{R,3}\phi_3(x) - \Sigma_{S,2\rightarrow 3}\phi_2(x) = 0$

Group 4: $-D_4 \frac{d^2}{dx^2}\phi_4(x) + \Sigma_{R,4}\phi_4(x) - \Sigma_{S,3\rightarrow 4}\phi_3(x) = 0$

Example 1: Four-group Diffusion Equations

$$M = \begin{bmatrix} -D_1 \frac{d^2}{dx^2} + \Sigma_{R,1} & 0 & 0 & 0 \\ -\Sigma_{s,1\to 2} & -D_2 \frac{d^2}{dx^2} + \Sigma_{R,2} & 0 & 0 \\ 0 & -\Sigma_{s,2\to 3} & -D_3 \frac{d^2}{dx^2} + \Sigma_{R,3} & 0 \\ 0 & 0 & -\Sigma_{s,3\to 4} & -D_4 \frac{d^2}{dx^2} + \Sigma_{R,4} \end{bmatrix} \qquad \phi = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \\ \phi_4(x) \end{bmatrix}$$

$$F = \begin{bmatrix} \chi_1 \nu \Sigma_{f,1} & \chi_1 \nu \Sigma_{f,2} & \chi_1 \nu \Sigma_{f,3} & \chi_1 \nu \Sigma_{f,4} \\ \chi_2 \nu \Sigma_{f,1} & \chi_2 \nu \Sigma_{f,2} & \chi_2 \nu \Sigma_{f,3} & \chi_2 \nu \Sigma_{f,4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{M}}\ \underline{\phi} = \frac{1}{k} \underline{\underline{F}}\ \underline{\phi}$$

Example 2: Five-group Diffusion Equation

Suppose you have a slab reactor with uniform material properties. Derive the five-group diffusion equation in matrix form assuming:

- There are 2 fast groups and 3 thermal groups
- The upper two groups have a fission source
- There is direct coupling between the fast groups
- Upscattering is allowed for the thermal groups
- Direct coupling applies to both downscattering and upscattering in the thermal groups

Example 2: Five-group Diffusion Equation

Group g	Thermal?	Fission Source?	Scattering to other groups
1	No	Yes	Down to 2
2	No	Yes	Down to 3
3	Yes	No	Down to 4
4	Yes	No	Down to 5, up to 3
5	Yes	No	Up to 4

Example 2: Five-group Diffusion Equations

$$M = \begin{bmatrix} -D_1 \frac{d^2}{dx^2} + \Sigma_{R,1} & 0 & 0 & 0 & 0 \\ -\Sigma_{s,1\to2} & -D_2 \frac{d^2}{dx^2} + \Sigma_{R,2} & 0 & 0 & 0 \\ 0 & -\Sigma_{s,2\to3} & -D_3 \frac{d^2}{dx^2} + \Sigma_{R,3} & -\Sigma_{s,4\to3} & 0 \\ 0 & 0 & -\Sigma_{s,3\to4} & -D_4 \frac{d^2}{dx^2} + \Sigma_{R,4} & -\Sigma_{s,5\to4} \\ 0 & 0 & 0 & -\Sigma_{s,4\to5} & -D_5 \frac{d^2}{dx^2} + \Sigma_{R,5} \end{bmatrix}$$

$$F = \begin{bmatrix} \chi_1 \nu\Sigma_{f,1} & \chi_1 \nu\Sigma_{f,2} & \chi_1 \nu\Sigma_{f,3} & \chi_1 \nu\Sigma_{f,4} \\ \chi_2 \nu\Sigma_{f,1} & \chi_2 \nu\Sigma_{f,2} & \chi_2 \nu\Sigma_{f,3} & \chi_2 \nu\Sigma_{f,4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \phi = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \\ \phi_4(x) \\ \phi_5(x) \end{bmatrix}$$

$$\underline{\underline{M}}\,\underline{\phi} = \frac{1}{k}\underline{\underline{F}}\,\underline{\phi}$$

Example 2: Five-group Diffusion Equations

$$
M = \begin{bmatrix}
-D_1 \frac{d^2}{dx^2} + \Sigma_{R,1} & 0 & 0 & 0 & 0 \\
-\Sigma_{s,1\to 2} & -D_2 \frac{d^2}{dx^2} + \Sigma_{R,2} & 0 & 0 & 0 \\
0 & -\Sigma_{s,2\to 3} & -D_3 \frac{d^2}{dx^2} + \Sigma_{R,3} & -\Sigma_{s,4\to 3} & 0 \\
0 & 0 & -\Sigma_{s,3\to 4} & -D_4 \frac{d^2}{dx^2} + \Sigma_{R,4} & -\Sigma_{s,5\to 4} \\
0 & 0 & 0 & -\Sigma_{s,4\to 5} & -D_5 \frac{d^2}{dx^2} + \Sigma_{R,5}
\end{bmatrix}
$$

What if direct coupling doesn't apply to downscattering?

$$
F = \begin{bmatrix}
\chi_2 \nu \Sigma_{f,1} & \chi_2 \nu \Sigma_{f,2} & \chi_2 \nu \Sigma_{f,3} & \chi_2 \nu \Sigma_{f,4} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad
\phi = \begin{bmatrix}
\phi_2(x) \\ \phi_3(x) \\ \phi_4(x) \\ \phi_5(x)
\end{bmatrix}
$$

$$
\underline{\underline{M}}\,\underline{\phi} = \frac{1}{k}\underline{\underline{F}}\,\underline{\phi}
$$

Example 2: Five-group Diffusion Equations

$$M = \begin{bmatrix} -D_1 \frac{d^2}{dx^2} + \Sigma_{R,1} & 0 & 0 & 0 & 0 \\ -\Sigma_{s,1\to2} & -D_2 \frac{d^2}{dx^2} + \Sigma_{R,2} & 0 & 0 & 0 \\ -\Sigma_{s,1\to3} & -\Sigma_{s,2\to3} & -D_3 \frac{d^2}{dx^2} + \Sigma_{R,3} & -\Sigma_{s,4\to3} & 0 \\ -\Sigma_{s,1\to4} & -\Sigma_{s,2\to4} & -\Sigma_{s,3\to4} & -D_4 \frac{d^2}{dx^2} + \Sigma_{R,4} & -\Sigma_{s,5\to4} \\ -\Sigma_{s,1\to5} & -\Sigma_{s,2\to5} & -\Sigma_{s,3\to5} & -\Sigma_{s,4\to5} & -D_5 \frac{d^2}{dx^2} + \Sigma_{R,5} \end{bmatrix}$$

What if direct coupling doesn't apply to downscattering?

$$F = \begin{bmatrix} \chi_2\nu\Sigma_{f,1} & \chi_2\nu\Sigma_{f,2} & \chi_2\nu\Sigma_{f,3} & \chi_2\nu\Sigma_{f,4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_2(x) \\ \phi_3(x) \\ \phi_4(x) \\ \phi_5(x) \end{bmatrix}$$

$$\underline{\underline{M}}\,\underline{\phi} = \frac{1}{k}\underline{\underline{F}}\,\underline{\phi}$$

Homework 9 Tips

- For the two-group diffusion equation problems, assume the core is bare and use familiar geometric buckling terms
  - We've done the first few steps
- For the four-group diffusion equation example, write down what assumptions about scattering you're working with.
  - There's no direct coupling assumption!

Homework Q/A

Problem 1) Calculating Power Density

$$Power\ Density\ at\ \vec{r} \equiv p''' = E_f \Sigma_f \phi(\vec{r}) \quad \left[\frac{w}{cm^3}\right]$$

- Remember, the diffusion equation gives us the flux shape without a magnitude

$$\phi = \boldsymbol{A} J_0\left(\frac{2.405}{\tilde{R}} r\right) \cos\left(\frac{\pi}{\tilde{H}} z\right)$$

- How do we find A?
  - You must scale the flux using the total power of the reactor
  - This requires integrating power density over the volume of the reactor – review examples in discussion 7

Problem 2) Reflector Savings

$$Reflector\ Savings \equiv R_{critical}^{bare} - R_{critical}^{reflected}$$

- Without a reflector, your criticality condition is just $B_g^2 = B_m^2$
- With a reflector, you have to derive it
  - Use the interface BC condition
  - You'll need to use Fick's law to get the current
  - See example at end of Discussion 8

Problem 3) Reflector Savings

- You can reuse the criticality condition from problem 2
- Part e) requires you to, once again, normalize the flux with the power
- Some helpful definitions for part f)

$$PPF \equiv \frac{\phi_{max}}{\phi_{avg}}, \qquad where\ \phi_{ave} = \frac{1}{V}\int_V \phi dV$$

Problem 4) Reflector Savings

- Unity means 1!
  - Thus, when the core material is infinite, $k_\infty > 1$
  - When the breeding blanket is infinite, $k_\infty < 1$
  - This means using different general solutions (trig vs exp) for different regions (see: useful_formulae.pdf on bcourses)
- Get as far as you can with simplifying the criticality condition
  - Setting up the problem is the most important

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$