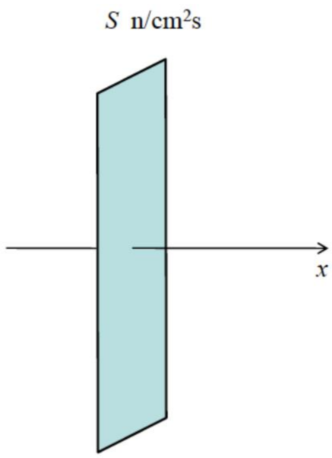
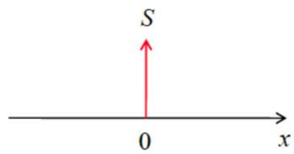
Diffusion Equation – Source) An infinite planar source, emitting S neutrons/cm^2s is placed at $x = 0$ in an infinite moderator with known properties (D, L). Derive the flux and current as a function of a distance from the source.

$$\frac{1}{v}\frac{\partial \phi(\vec{r},t)}{\partial t} = S(\vec{r},t) - \Sigma_a(\vec{r})\phi(\vec{r},t) + \nabla \cdot D(\vec{r})\nabla\phi(\vec{r},t)$$

Solution)





$$\phi''(x) - \frac{1}{L^2}\phi(x) + \frac{1}{D}\delta(x) = 0$$

BC: $\phi(x) \xrightarrow[x\to\pm\infty]{} 0$

$$\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = -\frac{1}{D}S\delta(x), \forall x$$

For $x > 0$:

$$\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = 0; \qquad B.C. \quad 1.\ \phi(x) \geq 0, \forall x; \quad 2.\ J(x) = \frac{S}{2}, x \to 0.$$

$x > 0$ $\qquad\qquad\qquad\qquad$ $x < 0$

$$\phi''(x) - \frac{1}{L^2}\phi(x) = 0 \qquad\qquad \phi''(x) - \frac{1}{L^2}\phi(x) = 0$$

$$\phi(x) = Ae^{-x/L} + Ce^{x/L} \qquad\qquad \phi(x) = Ae^{-x/L} + Ce^{x/L}$$

$$\phi(x) = Ae^{-x/L} \qquad\qquad \phi(x) = Ce^{x/L}$$

$$\phi(x) = Ae^{-|x|/L}$$

Use the second Boundary Condition – the Source Condition:

$$J(x) = -D\frac{d\phi(x)}{dx} = \frac{DA}{L}e^{-x/L} = \frac{S}{2} \quad \Rightarrow \quad A = \frac{SL}{2D}$$

$$\phi(x) = \frac{SL}{2D}e^{-|x|/L}$$

$$J(x) = \frac{S}{2}e^{-|x|/L}$$

Consider a bare sphere made of a uniform neutron multiplying material that is critical. Derive the shape of the flux $\phi(r)$ as a function of radius.

Solution)

$$\frac{1}{v}\frac{\partial\phi(\vec{r},t)}{\partial t} = S(\vec{r},t) - \Sigma_a(\vec{r})\phi(\vec{r},t) + \nabla\cdot D(\vec{r})\nabla\phi(\vec{r},t)$$

- Critical -> Steady state $(d\phi/\partial t = 0)$
- Uniform $(D,\ \Sigma_a$ constant)
- Multiplying $S(\vec{r},t) = \nu\Sigma_f\phi(\vec{r},t)$
- Slab (Spherical Cartesian)

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\phi}{dr} + B_m^2\phi(r) = 0$$

This equation has the general solution:

$$\phi(r) = C_1\frac{\cos(B_m r)}{r} + C_2\frac{\sin(B_m r)}{r}$$

BC 1) Finite flux

$$\lim_{r\to 0}\phi(r) < \infty \quad \text{but} \quad \lim_{r\to 0}\frac{\cos(B_m r)}{r} \to \infty \quad \text{so} \quad C_1 = 0$$

BC 2) Vacuum Boundary

$$\phi(\tilde{R}) = 0 = C_2\frac{\sin(B_m\tilde{R})}{\tilde{R}} \Rightarrow B_m\tilde{R} = n\pi,\ \ n = 0,1,2,\ldots$$

$$n = 0 \text{ is trivial, } \mathrm{n} > 1 \text{ gives negative flux, so } \mathrm{n} = 1 \text{ is real}$$

$$B_m = \frac{\pi}{\tilde{R}},\ \ B_m^2 = B_g^2 = \left(\frac{\pi}{\tilde{R}}\right)^2$$

Finally, our flux shape is

$$\phi(r) = C_2\frac{1}{r}\sin\left(\frac{\pi}{\tilde{R}}r\right)$$

Consider a reactor that is composed of a homogenous mixture of pure U-235 and graphite. Find the critical dimension if the reactor is:

a) A bare sphere

b) A bare finite cylinder with a height equal to twice the radius

Which of these reactor shapes has the smallest critical mass of U-235 and why?

$$\frac{N_C}{N_U} = 10^4, \qquad L^2 = 3040\ cm^2, \qquad \nu\sigma_f^U = 5.916b, \qquad \sigma_a^U = 2.844b,\ \ \rho^U = 19.1\ g/cm^3$$

$$\sigma_a^C = 3.4 \times 10^{-6}b, \qquad \rho^C = 1.60\ g/cm^3$$

Solution:

First, we calculate the material buckling for the provided material, since it will be the same for both reactors.

$$B_m^2 = \frac{k_\infty - 1}{L^2}$$

$$k_\infty = \varepsilon p \eta f \approx \eta f = \frac{\nu\Sigma_f^U}{\Sigma_a^U + \Sigma_a^C} = \frac{\nu\sigma_f^U}{\sigma_a^U + \frac{N_C}{N_U}\sigma_a^C} = \frac{5.916b}{2.844b + 10^4 \cdot 3.4 \times 10^{-6}b} = 2.06$$

$$B_m^2 = \frac{2.06 - 1}{3040\ cm^2} = 3.49 \times 10^{-4}\ cm^{-2}$$

a) We can relate the material buckling to the geometric buckling

$$B_m^2 = B_g^2 = \left(\frac{\pi}{R}\right)^2 => R = \sqrt{\frac{\pi^2}{B_m^2}} = 168.17\ cm$$

b) We do the same for cylinders:

$$B_m^2 = B_g^2 = \left(\frac{\pi}{2R}\right)^2 + \left(\frac{2.405}{R}\right)^2 = \left(\frac{\pi}{2R}\right)^2 + \left(\frac{4.81}{2R}\right)^2 = \frac{8.25}{R^2} => R = \sqrt{\frac{8.25}{B_G^2}} = 153.75\ cm$$

c) $m(sphere) < m(cylinder)$. You can show this by calculating the volume or mass, but you can intuitively know this by recognizing that the sphere will have the smallest amount of leakage.

Consider a critical bare slab of thickness a. Determine the flux peaking factor (maximum flux-to-average flux ratio). The flux shape is given by:

$$\phi(x) = Acos\left(\frac{\pi}{\tilde{a}}x\right)$$

Solution

$$PPF = \frac{\phi_{max}}{\bar{\phi}}$$

The maximum flux will be in the center of the slab:

$$\phi_{max} = \phi(x = 0) = Acos\left(\frac{\pi}{\tilde{a}}0\right) = A$$

The average flux is determined by integrating the flux over x. Normally we'd divide by V, but the slab is one dimensional:

$$\bar{\phi} = \frac{1}{a}\int_{-a/2}^{a/2}\phi(x)dx = \frac{A}{a}\int_{-a/2}^{a/2}cos\left(\frac{\pi}{\tilde{a}}x\right)dx = \frac{A}{a}\frac{\tilde{a}}{\pi}\left[\sin\left(\frac{\pi a}{2\tilde{a}}\right) - \sin\left(-\frac{\pi a}{2\tilde{a}}\right)\right]$$

$$\text{assuming a} \approx \tilde{\text{a}}$$

$$\bar{\phi} = \frac{A}{\pi}\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right] = \frac{A}{\pi}[1 - -1] = \frac{2A}{\pi}$$

Multi-group diffusion) Starting from a general steady-state multigroup neutron diffusion equation in slab geometry, derive four-group diffusion equations assuming that:

1) The fission source exists in the upper three groups
2) Only the lowest group contains thermal neutrons

$$\frac{1}{v_g}\frac{\partial \phi_g}{\partial t} = \chi_g \sum_{g'=1}^{G} \nu_{g'}\Sigma_{f,g'}\phi_{g'} + \sum_{g'=1}^{G} \Sigma_{s,g'\to g}\phi_{g'} - \Sigma_{tot,g}\phi_g + D_g\nabla^2\phi_g$$

No upscattering since there is only one thermal neutron group, and down-scattering is not assumed to be coupled.

$$M\phi = \frac{1}{k}F\phi$$

$$M = \begin{bmatrix} -D_1\dfrac{d^2}{dx^2} + \Sigma_{R,1} & 0 & 0 & 0 \\ -\Sigma_{s,1\to 2} & -D_2\dfrac{d^2}{dx^2} + \Sigma_{R,2} & 0 & 0 \\ -\Sigma_{s,1\to 3} & -\Sigma_{s,2\to 3} & -D_3\dfrac{d^2}{dx^2} + \Sigma_{R,3} & 0 \\ -\Sigma_{s,1\to 4} & -\Sigma_{s,2\to 4} & -\Sigma_{s,3\to 4} & -D_4\dfrac{d^2}{dx^2} + \Sigma_{R,4} \end{bmatrix}$$

$$F = \begin{bmatrix} \chi_1\nu\Sigma_{f,1} & \chi_1\nu\Sigma_{f,2} & \chi_1\nu\Sigma_{f,3} & \chi_1\nu\Sigma_{f,4} \\ \chi_2\nu\Sigma_{f,1} & \chi_2\nu\Sigma_{f,2} & \chi_2\nu\Sigma_{f,3} & \chi_2\nu\Sigma_{f,4} \\ \chi_3\nu\Sigma_{f,1} & \chi_3\nu\Sigma_{f,2} & \chi_3\nu\Sigma_{f,3} & \chi_3\nu\Sigma_{f,4} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \phi = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \\ \phi_4(x) \end{bmatrix}$$